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ANALYSIS OF THE ELECTROMAGNETIC EFFECTS OF A NUCLEAR EXPLOSION IN THE LOWER ATMOSPHERE

FINAL REPORT

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FOREWORD

This report consists of seven parts, originally written so that each could stand alone as a separate article or technical note. Each part contains its own abstract, which is placed at the start of that part. Likewise, a detailed table of contents is placed at the start, and references at the end of each part.

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Partition

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Part I. Electromagnetic Radiation from an Atmospheric Nuclear Detonation . . . . . . Roger E. Clapp

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ABSTRACT

From measured waveforms it has been determined that the electromagnetic pulse from a nuclear detonation is associated with the gradual establishment of a large dipole moment, which remains suspended in the atmosphere at the conclusion of the EMP waveform. From the gradual growth and the large magnitude, it is inferred that the source-current region is large, and that retardation across this source-current region is an important aspect of the EMP phenomenon. Retardation is also important in individual processes, since the primary Compton electrons are relativistic, and the forward directivity of their radiated electromagnetic fields can be attributed to retardation and can be encompassed in a distributed-current picture which allows correctly for retardation. When retardation is properly incorporated, it is found that for early times there is a momentary radial electric field projected above the detonation center, with the polarity to drive secondary electrons upward, thereby contributing substantially to the EMP dipole moment. The AFWL non-causal solution gives the wrong polarity for this early-time radial field, but the non-causal parts of the AFWL program can be replaced by a causal iteration procedure, without affecting the parts of the program which deal with electron attachment and air chemistry, and with the gamma rays and the primary source current.

1. INTRODUCTION

Previous reports in this series[1,2,3] have dealt with three aspects of the EMP (electromagnetic pulse) problem. There is first the phenomenological aspect. From measured waveforms it can be determined that a nuclear explosion in the lower atmosphere establishes a dipole moment in the air with a negative polarity. The detonation raises negative charge and leaves it suspended in the air. The time required for this process to take place is on the order of 100 microseconds, being longer for the larger-yield detonations, shorter for those of small yield.

The growth of the dipole moment is gradual, not abrupt. Also, detailed analysis of the measured waveforms (Chapter II of Reference 2) shows that the negative charge remains suspended at the end of the EMP waveform, decaying to zero only after a time which is very long in comparison with the duration of this very-low-frequency pulse. These two observations, based on the measured waveforms, lead to the inference that the source currents are not mainly confined to the near vicinity of the detonation point, but are distributed over distances measured in kilometers, at least in the vertical direction above the detonation center.

The second aspect of the EMP problem which has been dealt with in detail is the relativistic nature of the primary Compton currents which provide the driving source for the main EMP phenomena. Gamma rays from the nuclear detonation produce Compton electrons through collisions with air molecules. Because of the blockage by the ground under the detonation point, most of these Compton electrons are directed upward, and they thus contribute to the establishment of a negative dipole moment. At the same time, their motion (including their initial acceleration at the moment of ejection from an air molecule, and their more gradual deceleration as they are slowed by ionizing collisions with air molecules) will produce electric and magnetic fields which can act on other electrons in the vicinity. These other electrons include not only the primary Compton electrons but also secondary electrons produced by the ionizing collisions which slow the primary Compton electrons, and other secondary electrons released through the photoionizing action of X-ray, visible, and ultraviolet radiation emitted by the nuclear detonation.

Because of the relativistic velocities of the Compton electrons, there is a forward directivity in the electromagnetic fields that they generate. This forward

directivity is incorporated in the field expressions obtained from the Liénard-Wiechert potentials.[4] However, these potentials were in turn derived from a distributed-charge picture, in which the source electron was assumed to be a compact distribution of moving electrical charge. The retardation across this charge distribution then accounts for the forward directivity associated with the relativistically-moving electron. Thus the relativistic nature of the primary Compton currents can be incorporated directly, through the use of the Liénard-Wiechert fields, or indirectly, through the careful allowance for retardation across the full EMP source-current distribution.

The treatment of the secondary electrons comprises the third aspect of the EMP problem which has been considered in the previous reports in this series. These secondary electrons are produced by ionizations along the Compton tracks, and by other ionization processes such as the photoelectric ejection of electrons from air molecules by incident ultraviolet, visible, and X-ray photons. The secondary electrons, once released, will move in response to the local electromagnetic field, and will in turn make their contribution to this field distribution.

The field contributions, made by the secondary electrons in their movement in the local fields, will be retarded-field contributions, just as the fields generated by the Compton electrons were retarded fields. An integral equation results, whose solution gives the distribution of the secondary-electron currents. Before this integral equation can be written down in detail, it is necessary that expressions be given for the retarded fields associated with a particular source-current distribution. Once these expressions have been found, the integral-equation problem can be given explicitly, and solved by numerical or analytical methods.

There is a hidden hazard in attempts to solve Maxwell's equations by numerical methods which do not incorporate retardation explicitly. The hazard is attributable to the fact that Maxwell's equations admit an advanced-field solution in addition to the physically acceptable retarded-field solution. A general solution, therefore, will be a superposition of retarded and advanced solutions, unless special precautions are taken to exclude the advanced solutions from the beginning. As will be shown later, one numerical method which has actually been programmed at great expense violates this basic physical causality requirement. Fortunately, much of the program can be retained when the non-causal solution is converted to a causal solution.

2. GROWTH OF DIPOLE MOMENT

As determined from two successive integrations of an experimentally-measured waveform,[2] the dipole moment associated with the EMP source-current distribution has a time dependence which shows a gradual growth, on a time scale of the order of 100 microseconds. (In some cases there is an overshoot, but this also is gradual on the same time scale, and the attainment of the final dipole moment, after the moderate overshoot, coincides with the reduction of the vertical current flow to zero.) This 100-microsecond time scale can be compared with the one-microsecond time scale that characterizes the actual nuclear detonation.[5] It is apparent that the 100-microsecond radiated waveform, though initiated by the one-microsecond detonation, does not have its time dependence determined by the time scale of the detonation's chain reaction, but by some other phenomenon or phenomena.

After the prompt effects of the nuclear chain reaction, there are delayed reactions, in which gamma-rays are emitted from the detonation products. However, these delayed emissions decay with a time constant of the order of one microsecond, and will be relatively insignificant long before the lapse of 100 microseconds.

The prompt Compton currents, associated with the prompt gamma-rays from the nuclear chain reaction, will be confined in time to an interval associated with the detonation itself. However, the propagation time must be added, since the Compton ejections at a distance <u>r</u> from the detonation center will occur at a time which is later, by the time interval r/c, than the moment at which the ejecting gamma-rays were emitted by the nuclear reaction at $r = 0$. The attenuation in air of the pertinent gamma-rays leads to an attenuation with time which has about the same time constant, one microsecond, as the delayed gamma-ray emissions. Thus the gamma-ray propagation time will be inadequate to account for the slow growth of the dipole moment associated with the electromagnetic effects of a nuclear detonation which is in the lower atmosphere, near the ground.

It can be noted, however, that the range of the ultraviolet radiation from the detonation is substantially greater than the range of the high-energy gamma-rays responsible for the Compton ejections. Furthermore, the forward directivity of the ejected Compton electrons leads to a voltage pulse which is projected outward at the velocity of light, far beyond the the distance reached by the Compton electrons themselves. This voltage pulse has the polarity to drive secondary electrons in the same radial direction in which the Compton electrons were moving,

voltage
and in this way the/pulse has the effect of extending greatly the range to which the upward electron motion reaches. By extending the range of the vertical current, this hybrid phenomenon (radial electric field from inner Compton electrons, multiplied by conductivity from outer photoionization processes) provides a mechanism which can explain the observed time scale for the growth of the dipole moment associated with the EMP source-current distribution.

In addition there may be higher-order interaction processes, in which the secondary-electron motion at early times contributes fields which lead to secondary-electron motion at later times. The computation of these complicated relaxation processes must be left to analytic or numerical methods, and cannot readily be foreseen by way of general principles. For the computation to have physical significance, however, it is essential that retardation be incorporated correctly, so that effects will not precede their causes, and the propagation of electromagnetic fields will not be faster than the velocity of light.

3. RELATIVISTIC EFFECTS

In earlier reports[2,3] two source models were analyzed relativistically with the use of the field expressions obtained from the Liénard-Wiechert potentials.[4] One of these models, called the shell model, was carried to the point of waveform calculation. In this model, a spherical shell of electrons is ejected at one radial distance, moves outward at a relativistic velocity, and then is deposited at a larger radial distance. That is, the electrons composing this moving shell have all been ejected simultaneously from an inner spherical surface, all move radially outward at the same velocity, and are simultaneously deposited on an outer spherical surface. The inner shell is initially uncharged, so that when the negatively-charged electrons are ejected, they leave behind a stationary shell of positive ions.

Two different symmetries were considered. One was spherical symmetry, with no angular dependence of the density of electrons on the moving shell. The other was cosine asymmetry, with the electron density set proportional to the cosine of the polar angle on the spherical surface. (This makes the lower hemisphere positively charged, with positrons, or with the images of the upper electrons in a horizontal ground plane.)

For the case of spherical symmetry, the results of the relativistic calculations were identical with the results obtainable from Gauss's law of the electrostatic field. Of the six components of the electric and magnetic fields, only the radial component of the electric field does not vanish by symmetry. Furthermore, for an observation point outside the outer spherical surface, the magnitude of this radial component, E_r, vanishes at all times, when the contributions of the moving electrons, the positive ions left behind, and the stopped electrons that have reached the outer spherical shell, are all added together. These separate contributions, however, do not vanish individually.

It is instructive to examine the role played by retardation across the source region. Because of the finite propagation velocity of electromagnetic effects, an observer cannot be sure that the shell-model source is actually spherically symmetric until enough time has elapsed to permit him to receive signals from the charged particles on the far side of the source region.

Figure 1 shows an early stage in the development of the shell model. What is shown is the source distribution contributing to the field component E_r at a particular observation point, as the source would appear to that observer with retardation taken into account. At the early moment depicted here, the observer is receiving the fields emitted by a sector of moving electrons, on the near side of the shell-model source, and the electrostatic fields from the positive ions which they left behind on the inner spherical surface. It is too early for the observer to receive the electrostatic fields from any of the stationary negative ions that will be formed when the moving electrons come to rest on the outer spherical surface.

Figure 2 shows a somewhat later stage. The nearer electrons have come to rest, forming a sector of negative ions which is bounded by the dashed lines in the figure. A band of moving electrons is also visible, and the exposed positive ions on the inner spherical surface now cover a sector which is greater than a hemisphere.

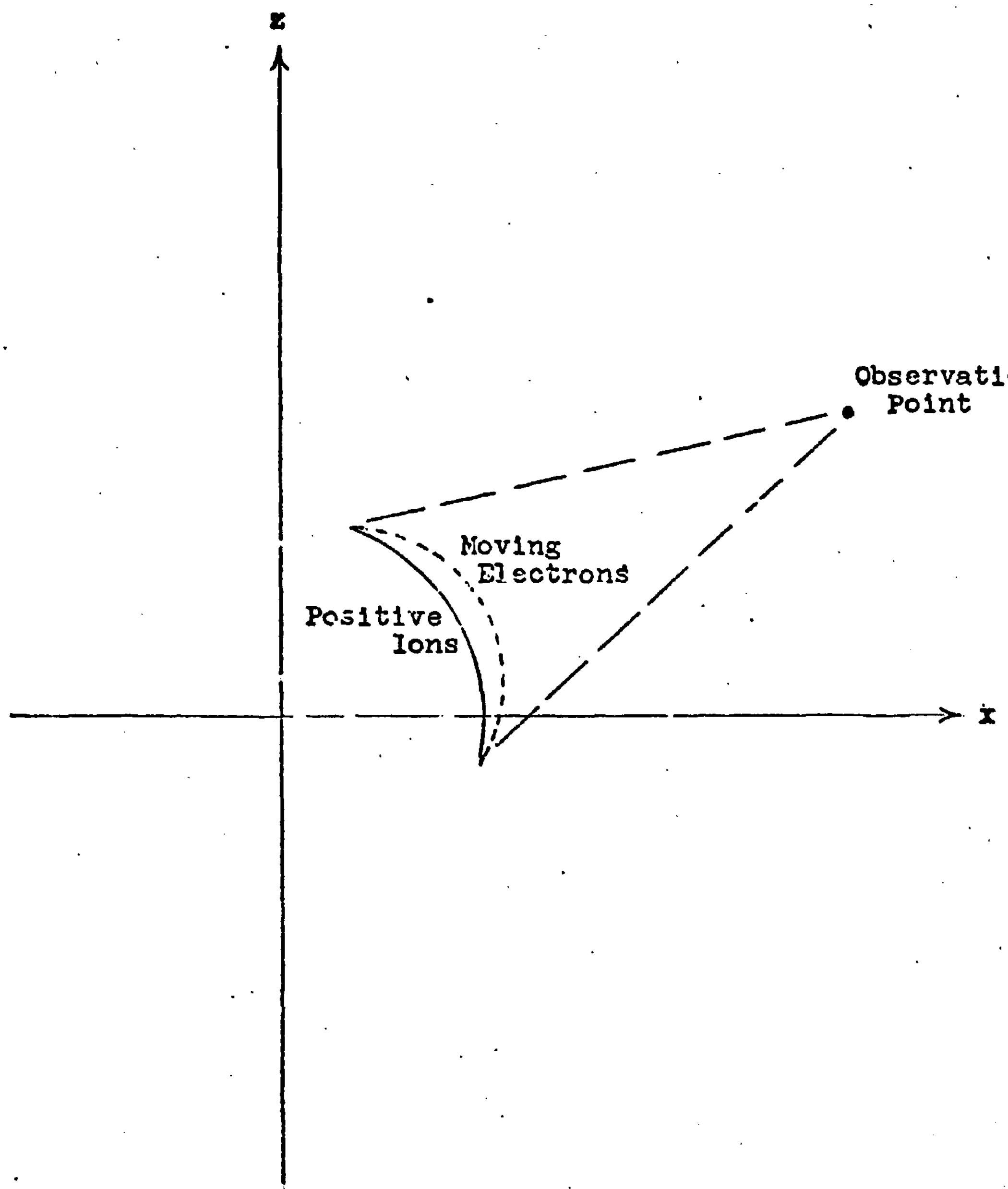

Figure 1. The shell model, as seen from the observation point at an early instant, before any of the stopped electrons (negative ions) can be seen.

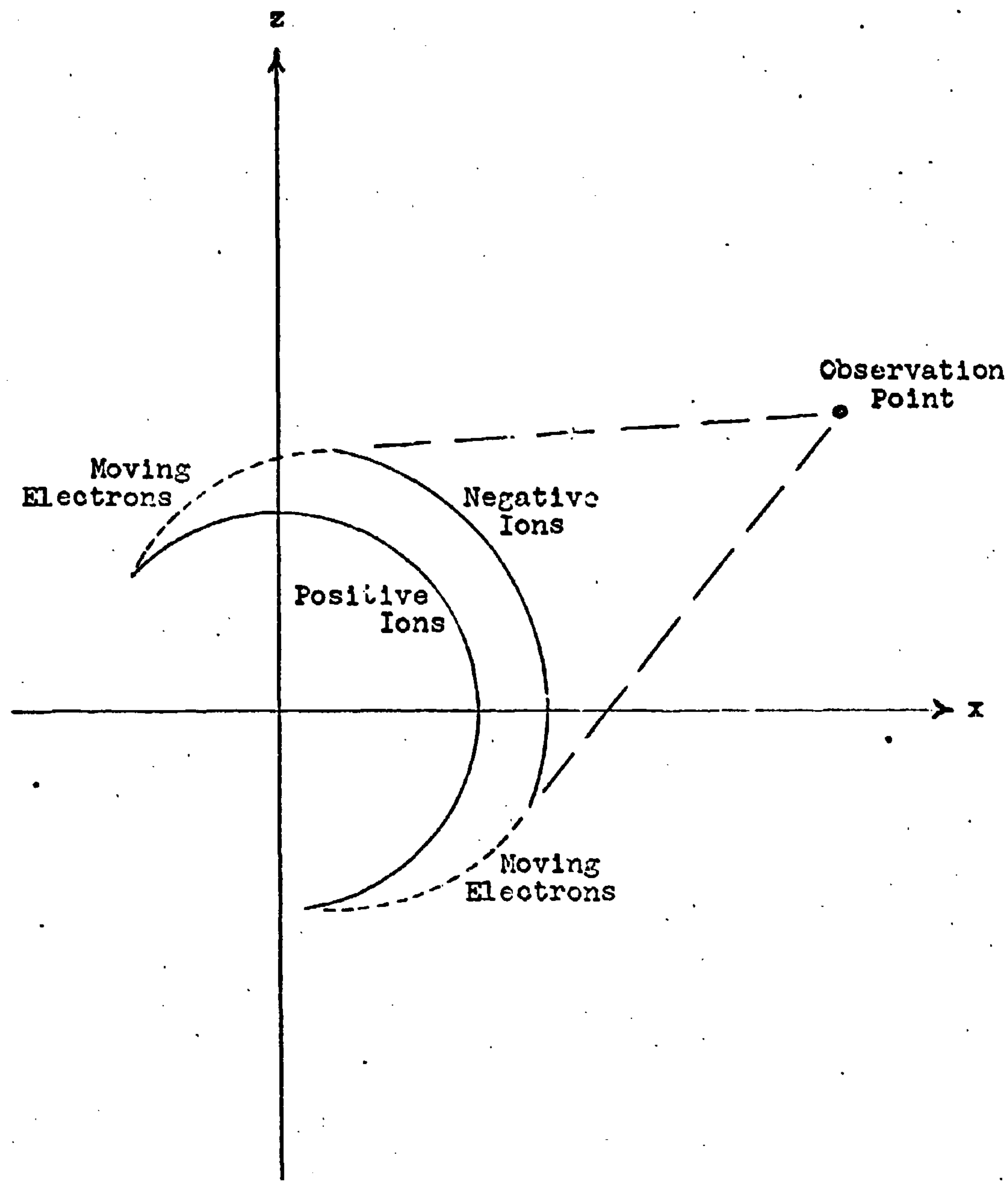

Figure 2. The shell model, as seen from the observation point at a moment somewhat later than the moment shown in Fig. 1. Some of the electrons have reached the outer shell, forming negative ions, and more than half of the positive ions on the inner shell have been exposed.

Figure 3 is a still later stage. The full inner shell of positive ions is exposed, and most of the ejected electrons have reached the outer shell and have come to rest as negative ions. A small sector of moving electrons remains visible, but these will shortly reach the outer shell and stop there.

It should be emphasized that the time scale for these three figures represents the observer's time scale, and the source distributions are those that he would sense. The appearance of asymmetry is solely due to the differences in propagation time from different parts of the source to the observation point. The source distribution itself is actually spherically symmetric in this instance.

For this shell model the separate contributions of the positive and negative ions and the moving electrons, including the fields generated by the ejection and deceleration processes, can all be evaluated in closed form. The moving electrons are treated relativistically. When the contributions are added together, the resulting magnitude for E_r at an observation point outside the source region is found to be exactly zero at all times.

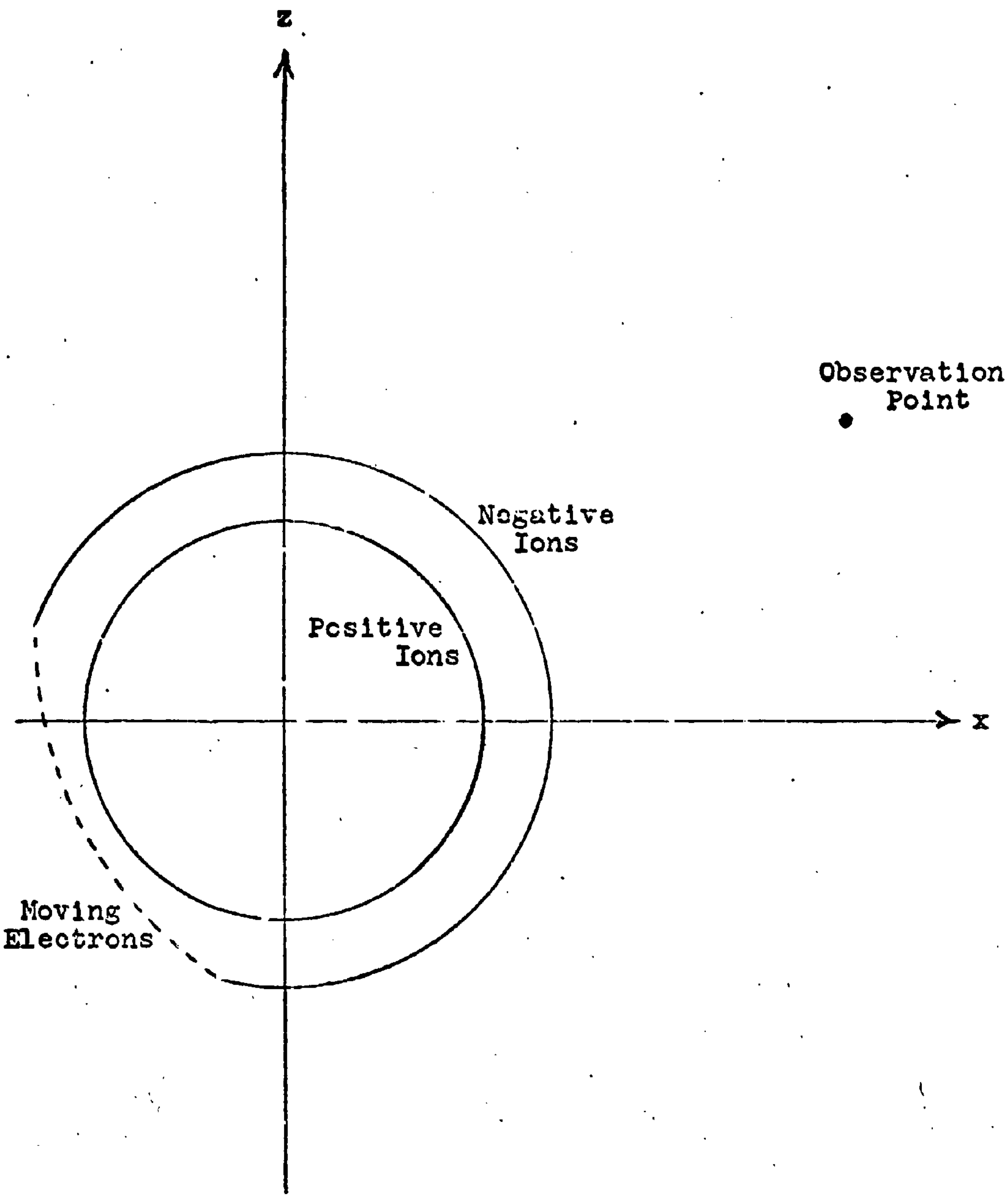

Figure 3. The shell model at a late stage. All of the positive ions on the inner shell have been exposed, and most of the electrons have reached the outer shell and stopped there to form negative ions. Because of the retardation, a few of the moving electrons on the far side of the source remain visible at the observation point.

If spherical symmetry is relinquished, then there is no longer this exact cancellation of the different contributions to the radial electric field, E_r, at an observation point outside the shell-model source distribution. Also, there is no longer the limitation of the generated electromagnetic field components to the single component E_r. In particular, if a weight function is introduced, weighting the currents by the factor cosine θ; where θ is the polar angle measured down from the positive z-axis (assumed to be vertically upward), then the result is a source model which will be called the shell model with cosine asymmetry.

A closely related model is the opposed-hemisphere shell model. For this model, the source currents can be considered to be the upper half of the shell model with spherical symmetry, together with the image currents which would accompany such a hemispherical shell source if it were located directly above a perfectly conducting ground plane. Figure 4 shows the distribution of positive and negative ions in the opposed-hemisphere shell model, after the electrons ejected from the inner shell (and their positively-charged images) have been deposited on the outer spherical shell.

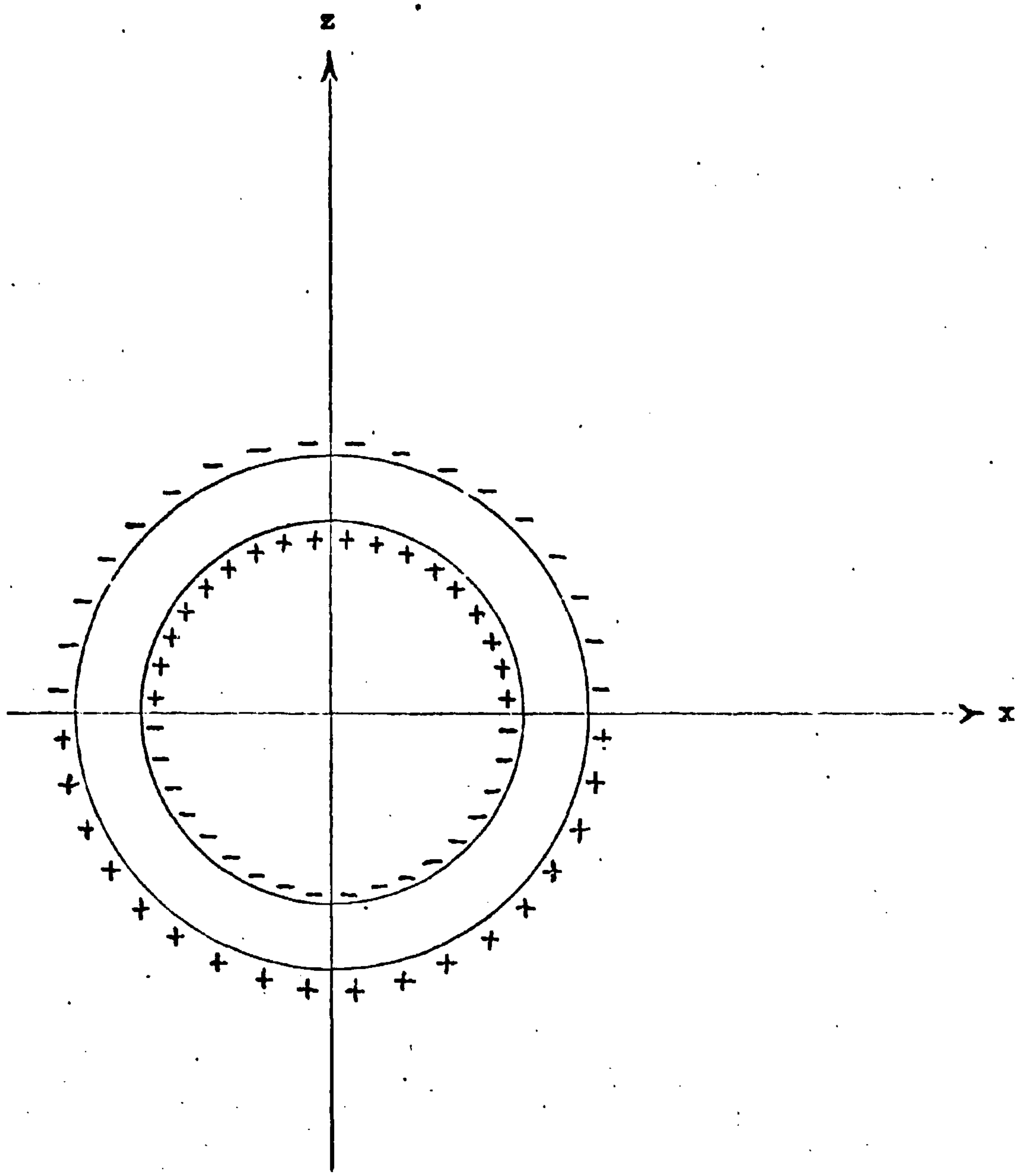

Figure 5. The opposed-hemisphere shell model, after the ejected electrons (and their positively-charged images) have traveled from the ejection points on the inner spherical shell to the deposition points on the outer spherical shell.

When the opposed-hemisphere weight function is expressed as a superposition of spherical-harmonic modes, involving weight factors which are Legendre polynomials $P_n(\cos\theta)$, then the leading term has the weight function for the shell model with cosine asymmetry. This weight function is $P_1(\cos\theta)$, which is just equal to the cosine of θ. For the shell model with cosine asymmetry, Figure 5 shows the ion distribution after the moving electrons have come to rest on the outer spherical surface.

When the shell model with cosine asymmetry is examined in detail, it is found that the time history of the fields at an observation point outside the outer shell can be interpreted with the aid of Figures 1-3. However, the weight function, cosine θ, enters as a factor multiplying the distributions of positive ions, moving electrons, and negative ions. The symmetry about the z-axis remains in this model, but the loss of spherical symmetry means that there are three nonvanishing field components: E_r, E_θ, and H_ϕ. The other three field components, E_ϕ, H_r, and H_θ, vanish as a consequence of the symmetry about the z-axis and the restriction of the electrons (and their images) to radial motion only.

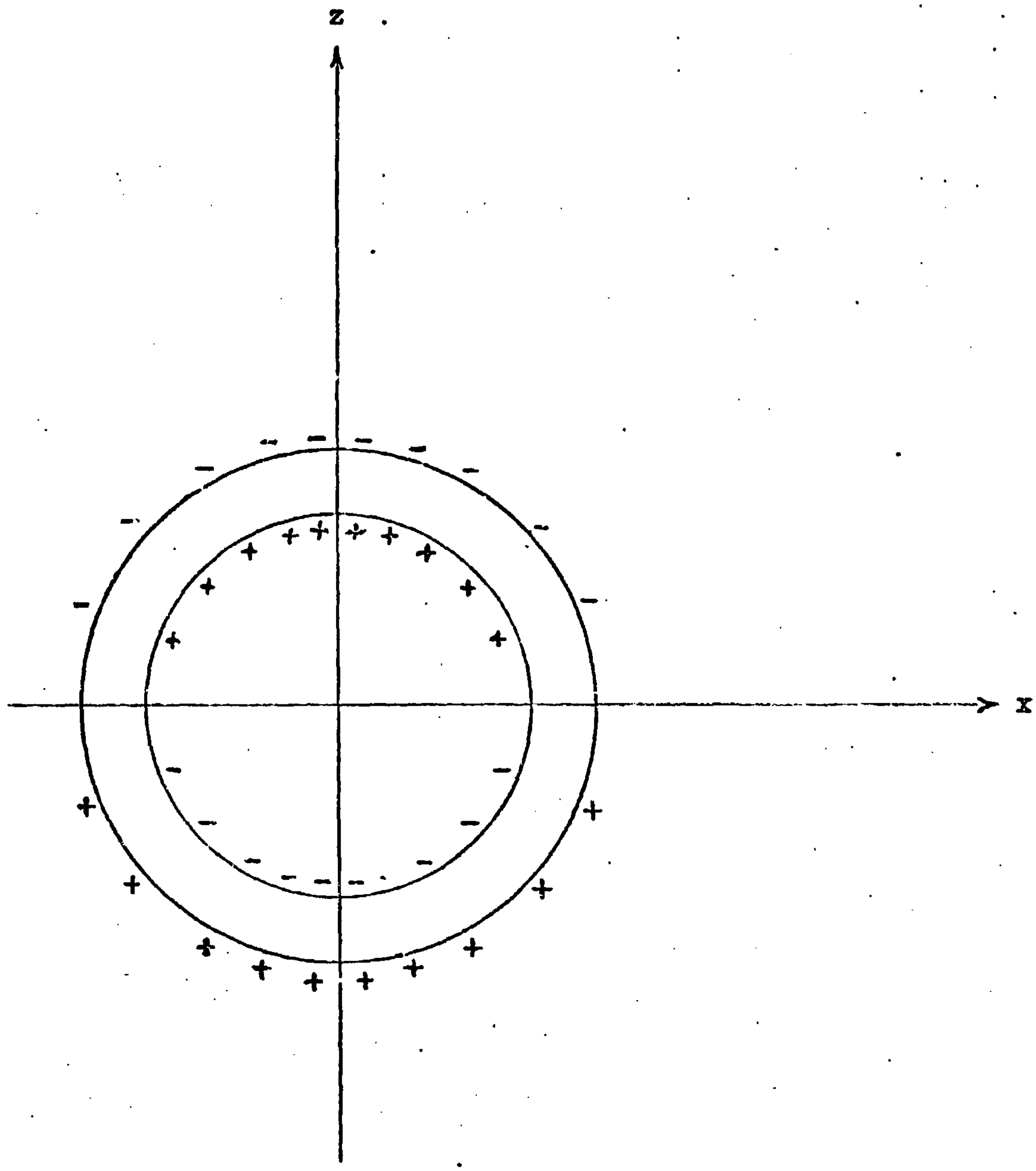

Figure 15. The shell model with cosine asymmetry, after the ejected electrons (and their positively-charged images) have traveled from the ejection points on the inner spherical shell to the deposition points on the outer spherical shell.

As in the case of spherical symmetry, the field contributions of the ions and electrons, at an outside observation point, can be evaluated in closed form. The result for the case of E_r has been given explicitly in Reference 3. When the contributions were added together, it was found that for an observation point above the source (on the positive z-axis at a position above the outer shell), the radial electric field rapidly approached a negative peak, then more slowly decayed back to a residual value associated with the static dipole moment of the charge distribution in Figure 5. The polarity of E_r was at all times negative, in a direction to drive secondary electrons upward if there had been any secondary electrons present, in the region above the shell-model source distribution.

For an observation point located along the z-axis, the field components E_θ and H_ϕ vanish by symmetry. However, for an observation point located in the neighborhood of the ground plane, to one side of the source, E_θ and H_ϕ have their maximum values while E_r vanishes. For an intermediate location, as in Figures 1-3, all three components are nonvanishing at the observation point.

While the electric field components, E_r and E_θ, are found to have residual magnitudes associated with the dipole moment of the final ion distribution (Figure 5), the magnetic field component, H_ϕ, is found to be directly associated with the moving electrons that are indicated in Figures 1-3 (as modified by the cosine weight function). The magnetic field is zero at the observation point up until the moment when the first of the moving electrons becomes 'visible' at this observation point, and is zero again for times later than the moment at which the farthest moving electron comes to rest on the outer shell, as sensed at the observation point with retardation appropriately incorporated. These moments marking the beginning and ending of the magnetic-field waveform are defined as the initial moment, slightly preceding the moment depicted in Figure 1, and the final moment, just after the moment depicted in Figure 3.

For times which lie between the initial and final moments, the magnetic field at the observation point can be described as the sum of two terms. One of these terms, the induction term, falls off with radial distance in proportion to r^{-2}. The other term, the radiation term, falls off with radial distance in proportion to r^{-1}. Each term vanishes separately for times which lie outside the interval between the initial and final moments, defined above.

4. RETARDATION AND CAUSALITY

The shell-model calculations described in the previous Section treated the moving electrons as relativistic particles, and utilized the Liénard-Wiechert field expressions.[4] The same waveform expressions can be obtained from a distributed-source-current picture, replacing the moving-charged-particle picture, provided that the current distribution is inserted into an integral formulation of the fields[6] in which retardation is accurately incorporated.

When this integral method is used, the shell-model current distribution, when reduced from four-dimensional space-time to the two-dimensional (r,t)-plane, has the form of an elongated delta-function, as illustrated in Figure 6. For this reduction, the angular dependence of the fields and the current has been expressed in terms of the appropriate vector spherical harmonics[6], and only the dependence upon radial distance, r, and time, t, remains in the reduced problem.

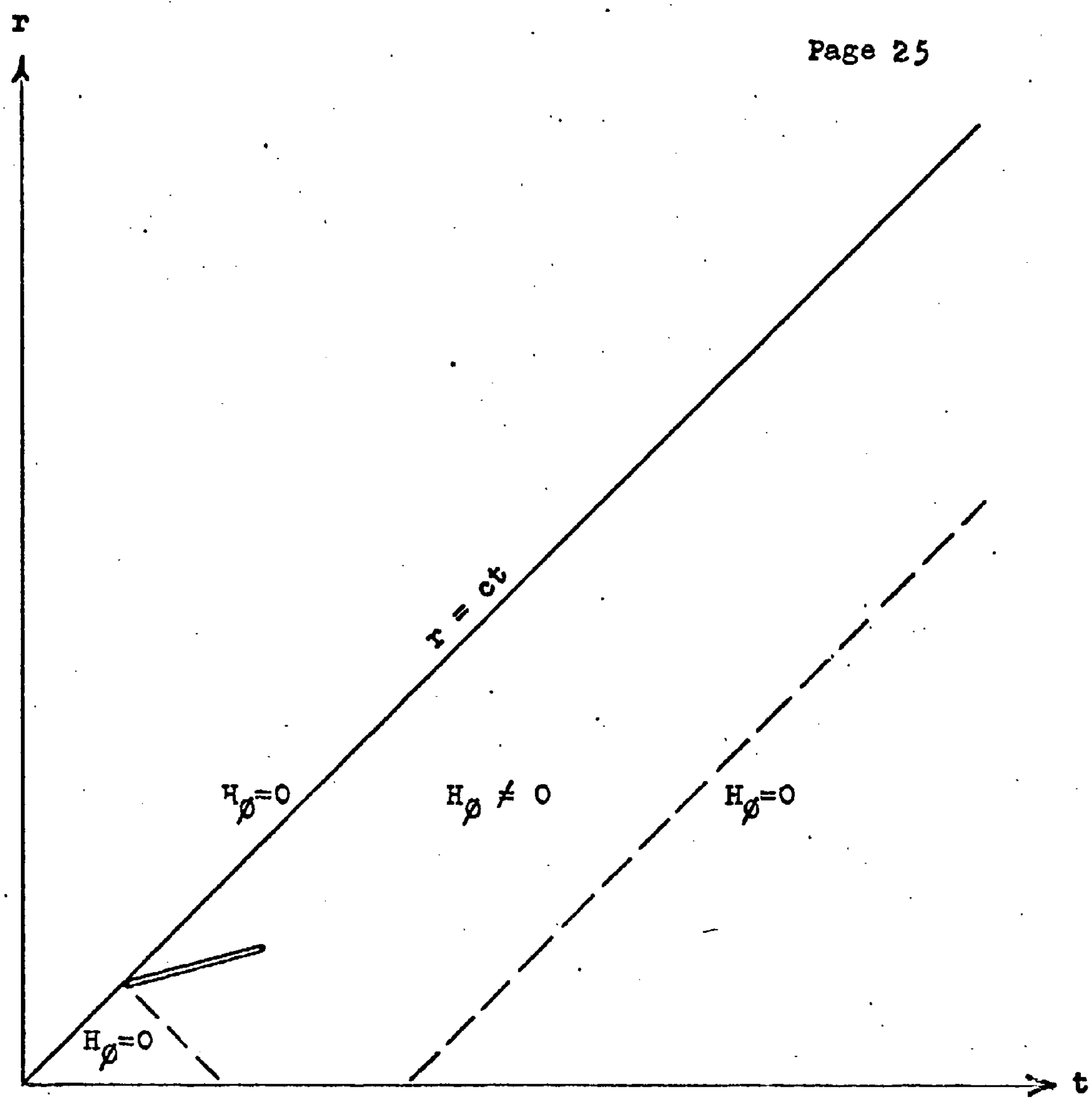

Figure 36. The problem domain, in the (r,t)-plane, for the shell model with cosine asymmetry. The source current is an elongated, tilted delta-function, shown here as an elongated oval. The region in which the magnetic field differs from zero is bounded by the two dashed lines, and by portions of the lines $r = 0$ and $r = ct$.

The choice of the origin of time, in Figure 6, has been made in such a way that the ejection of the electrons from the inner shell could have been through Compton processes involving gamma rays which left the point $r=0$ at the time $t=0$. Thus the ejections take place at a point in the diagram which lies on the line $r=ct$, and the current flow extends to the right of this line. The slope of the elongated delta-function is equal to the ratio v/c, where v is the velocity of the moving electron in the shell-model picture, and c is of course the velocity of light.

Figure 6 shows the region where the magnetic field, H_ϕ, differs from zero. For an observation point at a particular radial distance, r, outside the source region, a horizontal section of Figure 6 shows that the magnetic field will differ from zero for a finite time interval. This is the interval during which moving electrons are 'visible' at that observation point.

In the integral method[6] the field components at a space-time point P are expressed as explicit integrals over the causally accessible source-current distribution. Through separation into vector-spherical-harmonic modes,

the problem is reduced to two dimensions, and the field components for a particular mode are expressed as integrals in the (r,t)-plane, over the current components for the same mode.

The realm of this integration in the (r,t)-plane is shown in Figure 7. It is assumed that no source currents flow in the region to the left of the diagonal line $r = ct$, because it is the EMP problem that is to be treated, and the nuclear detonation initiates the source-current flow. The currents which contribute to the field components at a space-time point P are then the currents in the shaded regions in Figure 7. The horizontal shading indicates the currents which can make inductive and radiative contributions to the field components at P. (Thus the magnetic field components at P must be generated by currents in the region that is horizontally shaded.) In addition, there are electrostatic contributions by the currents in the rectangular region with horizontal shading, and also by the currents in the triangular region with vertical shading.

It is the limitation imposed by the causality diagram of Figure 7, which accounts for the limited region of nonzero magnetic field shown in Figure 6, from the point of view of the integral method.

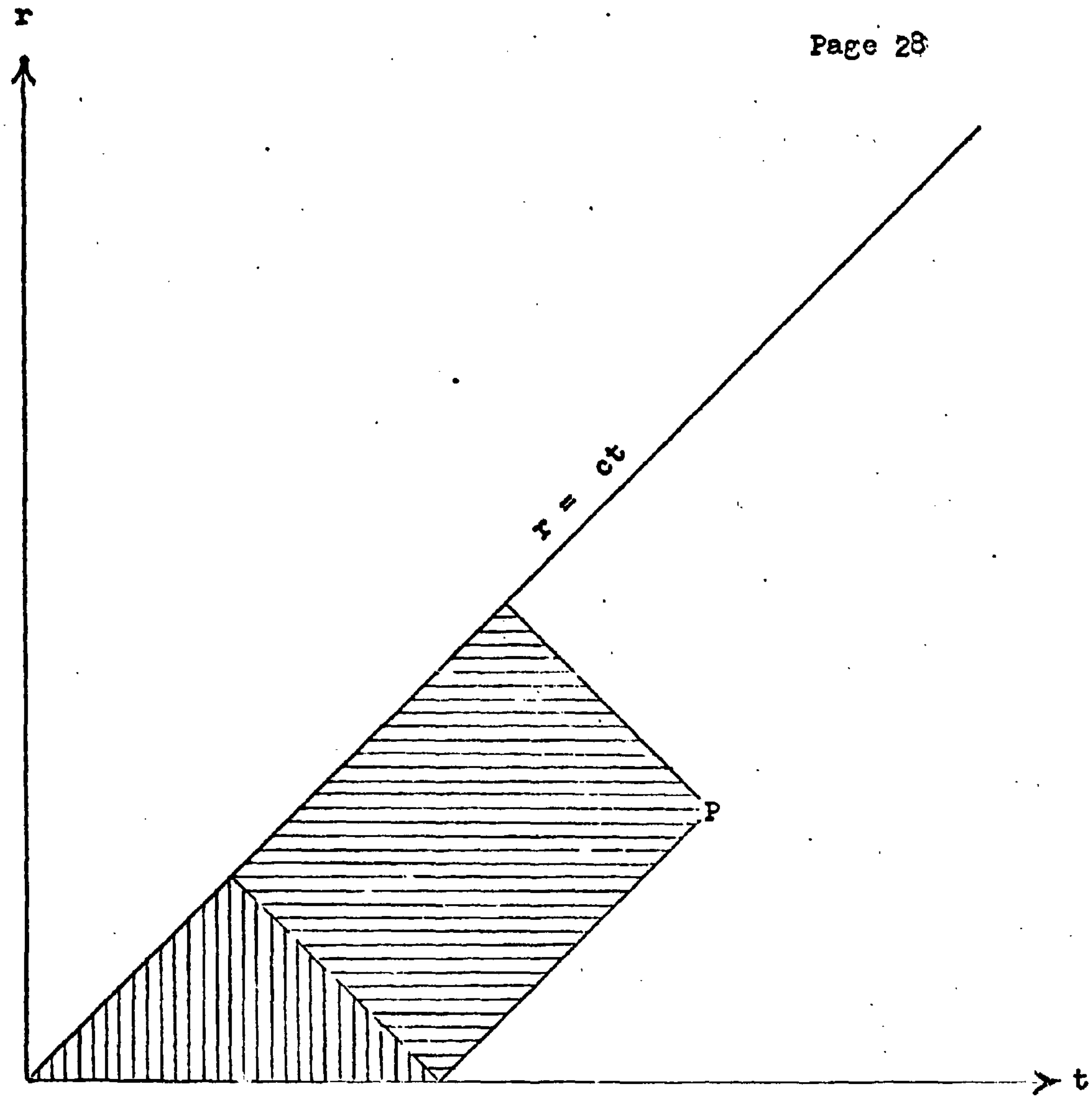

Figure 7. The causality diagram for the integral method. The field components at the space-time point P are determined by the currents in the shaded region. The inductive and radiative contributions are made by the currents in the rectangular region shown with horizontal shading. The triangular region with vertical shading produces only electrostatic-field contributions.

5. SECONDARY ELECTRONS

The treatment of the secondary electrons comprises the third of the three aspects of the EMP problem that were referred to in the Introduction. Some of the secondary electrons are those that are released by the ionizing collisions that serve to slow the Compton electrons in their passage through the air. Other secondary electrons are released from air molecules through the photoelectric ionization processes in which the energy for ionization is provided by X-rays, ultraviolet light, and visible light from the nuclear detonation. The secondary electrons released by ionizations along the Compton tracks will be mainly confined to the inner regions where there are many Compton-electron-producing gamma rays. The X-ray ionization will also have a relatively short range. However, the ionization which is produced by ultraviolet light will be spread over a large volume, because of the relatively long range of ultraviolet photons in air, as compared with X-rays and gamma rays.

The secondary electrons, once released by whatever ionizing process, will move in response to the local electric field strength. Their motion, in turn, will constitute a current which will generate its own electric and magnetic fields. Since all of the processes are transient, and since there are propagation delays involved each time a moving electron affects another electron, it is possible in principle to set up a mathematical iterative procedure, beginning at very early times when the secondary ionization is small. For the initial steps of the iteration procedure, what matters is that the product of the conductivity associated with the secondary ionization, and the electric field associated with the primary Compton current, should give a secondary current which is small in comparison with the primary Compton current. The iteration[7] then proceeds by steps in the (r,t)-plane which are small enough so that numerical instabilities are avoided and a smooth, gradual function for the net current (the sum of the Compton current and the secondary current) is obtained.

It is also possible to call upon analytical methods for the solution of the integral equation, which gives the motion of the secondary electrons and the fields they generate.[7] In regions of space-time where the secondary conductivity is not very great, an analytical iteration procedure can be used, where the secondary current is written as a sum of terms. The first term is the product of the conductivity function and the electric field generated by the Compton electrons alone. The second term is the product of the conductivity function and the electric field generated by the first term alone, and so forth. This approach should be useful for early times and for large radial distances.

In regions of space-time where the secondary conductivity is very high, the above iteration procedure can lead to numerical oscillation or divergence. Here the straightforward numerical approach can be used, but an alternative is one in which the net current or the net local electric field is expanded in a finite set of linearly independent functions which span the region of interest in the (r,t)-plane. The integral equation can then be made to yield a least-squares condition on the coefficients in the above expansion.

6. CONCLUSIONS

Special emphasis has been laid on the causality requirement, in the solution of the EMP problem. The reason for this emphasis is that the source volume is large enough to make the retardation time across this volume a significant part of the generated electromagnetic signal waveform duration. A solution to Maxwell's equations which does not allow properly for this retardation may have little or no relationship to the physical processes which take place in the vicinity of a nuclear detonation.

It was noted, for example, that the shell-model calculation showed an early-time radial electric field which had the polarity to drive secondary electrons upward above the detonation center, and that this early-time radial field coincided in time with the long-range ionization by ultraviolet light from the detonation, and was therefore important in the analysis of the mechanisms contributing to the establishment of the large dipole moment which remains after the EMP waveform is over.

There is a non-causal numerical solution of Maxwell's equations which has been applied to this EMP problem.[7] As evidence that this non-causal solution does indeed give wrong answers, it has been noted[8] that the early-time radial electric field, obtained from this non-causal solution, has the wrong polarity. Instead of driving secondary electrons upward above the detonation center, the early-time radial electric field from the non-causal calculation drives secondary electrons downward. Thus an important physical process contributing to the dipole-moment establishment does not appear in this non-causal solution.

It is fortunate that the non-causal iteration method can be separated from the remainder of this numerical solution of Maxwell's equations, and can be replaced by a causal iteration procedure, without changing the portions of the program which deal with electron attachment and air chemistry, and with the gamma rays and the primary source current.

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6. See Parts II-VI of this Final Report.

7. See Part VII of this Final Report.

8. Private communication from Capt. John Darrah, Air Force Weapons Laboratory, Albuquerque, New Mexico.

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Part II. Generalized Addition Theorem for Spherical Harmonics . . . . Roger E. Clapp

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ABSTRACT

In analogy with the familiar addition formula for Legendre polynomials, a generalized addition theorem is proved. A general spherical harmonic, depending on the two angles θ_1 and ϕ_1, is expressed as an expansion involving spherical-harmonic functions of (θ, ϕ) and of (θ', ϕ'). The six angles are related to each other through the equations

$$\cos \theta_1 = \cos\theta \cos\theta' - \sin\theta \sin\theta' \cos(\phi' - \phi \cos\theta) ,$$

$$\sin\theta_1 \cos(\phi_1 - \phi) = \sin\theta \cos\theta' + \cos\theta \sin\theta' \cos(\phi' - \phi \cos\theta) ,$$

$$\sin\theta_1 \sin(\phi_1 - \phi) = \sin\theta' \sin(\phi' - \phi \cos\theta) .$$

The expansion is then used in the proof of an integral theorem for spherical harmonics.

1. INTRODUCTION

The familiar addition formula for Legendre polynomials[1] can be written as

$$P_n(\cos\theta') = P_n(\cos\theta_1)\,P_n(\cos\theta) + 2\sum_{\mu=1}^{\mu=n}\frac{(n-\mu)!}{(n+\mu)!}\,P_n^{\mu}(\cos\theta_1)\,P_n^{\mu}(\cos\theta)\cos\mu(\phi_1-\phi)\,, \qquad (1.1)$$

where

$$\cos\theta' = \cos\theta_1\cos\theta + \sin\theta_1\sin\theta\cos(\phi_1-\phi)\,. \qquad (1.2)$$

The angle θ' can be interpreted as the included angle between the two vectors $\underset{\sim}{r}_1$ and $\underset{\sim}{r}$, whose directions are specified by (θ_1,ϕ_1) and (θ,ϕ), respectively.

The Legendre polynomials are a subset of the more general spherical harmonics. Eq. (1.1) is a relationship between members of such a subset, on the left-hand side, and members of two full sets, on the right-hand side. It is to be expected that other similar relationships exist, in which a full set of spherical harmonics, involving θ' and a suitably

defined ϕ', is related to the two full sets of spherical harmonics involving (θ_1, ϕ_1) and (θ, ϕ).

It is first necessary to find a suitable definition for the angle ϕ' which is to accompany θ'. A simple definition is found, and the reason for its selection is explained. An equation of the form (1.1) is then given, in which $P_n(\cos\theta_1)$ is expressed as a summation over spherical harmonics which are functions of $\theta, \phi, \theta', \phi'$. Finally, this equation is generalized so that a general spherical harmonic, written as a function of θ_1, ϕ_1, is then given as a summation over spherical-harmonic functions involving $\theta, \phi, \theta', \phi'$.

2. DEFINITION OF ϕ'

The vectors $\underset{\sim}{r}$ and $\underset{\sim}{r}_1$, with polar-coordinate components (r,θ,ϕ) and (r_1,θ_1,ϕ_1), are illustrated in Figure 1. These vectors are shown with respect to a rectangular coordinate system, with axes labeled by $\underline{x}$, $\underline{y}$, and $\underline{z}$.

Figure 2 shows a new set of coordinate axes, labeled $\underline{x}'$, $\underline{y}'$, and $\underline{z}'$, which are obtained from the first set by two rotations. The first is a rotation about the z-axis by the angle ϕ; this rotation moves the y-direction into the position of the y'-axis, which thus must lie in the (x,y)-plane. The second rotation is about the y'-axis, and moves the z-direction down by the angle θ, until it lies in the position of the z'-axis, which lies along the vector $\underset{\sim}{r}$. In this way, by the two rotations through the angles ϕ and θ, the (x,y,z)-directions are moved into the positions shown as the (x',y',z')-axes.

Figure 2 thus defines the z'-axis unequivocally, since this axis must lie along the direction of the vector $\underset{\sim}{r}$. The y'-axis will also be defined uniquely, if θ is greater than zero and less than π. The plane through the origin of

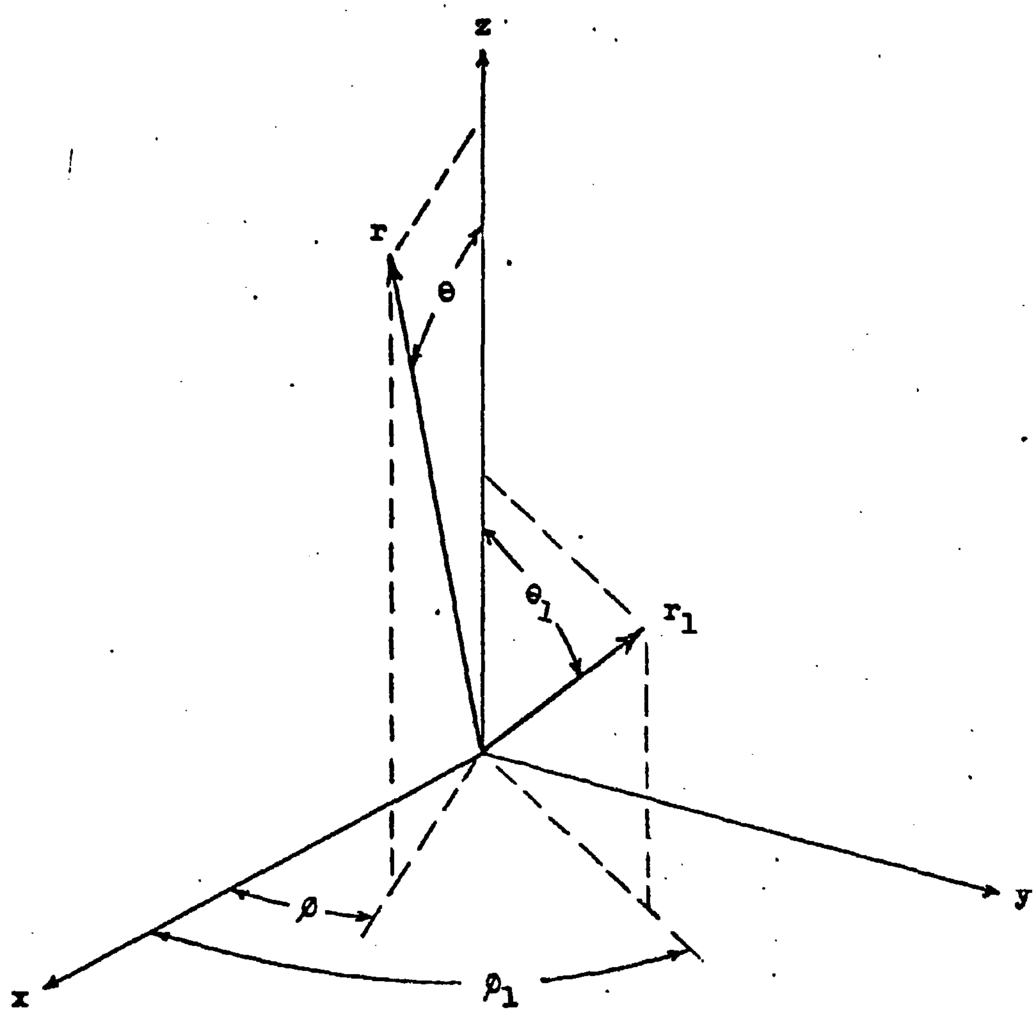

Figure 1. The vectors $\underset{\sim}{r}$ and $\underset{\sim}{r}_1$, shown in relation to the (x,y,z)-axes.

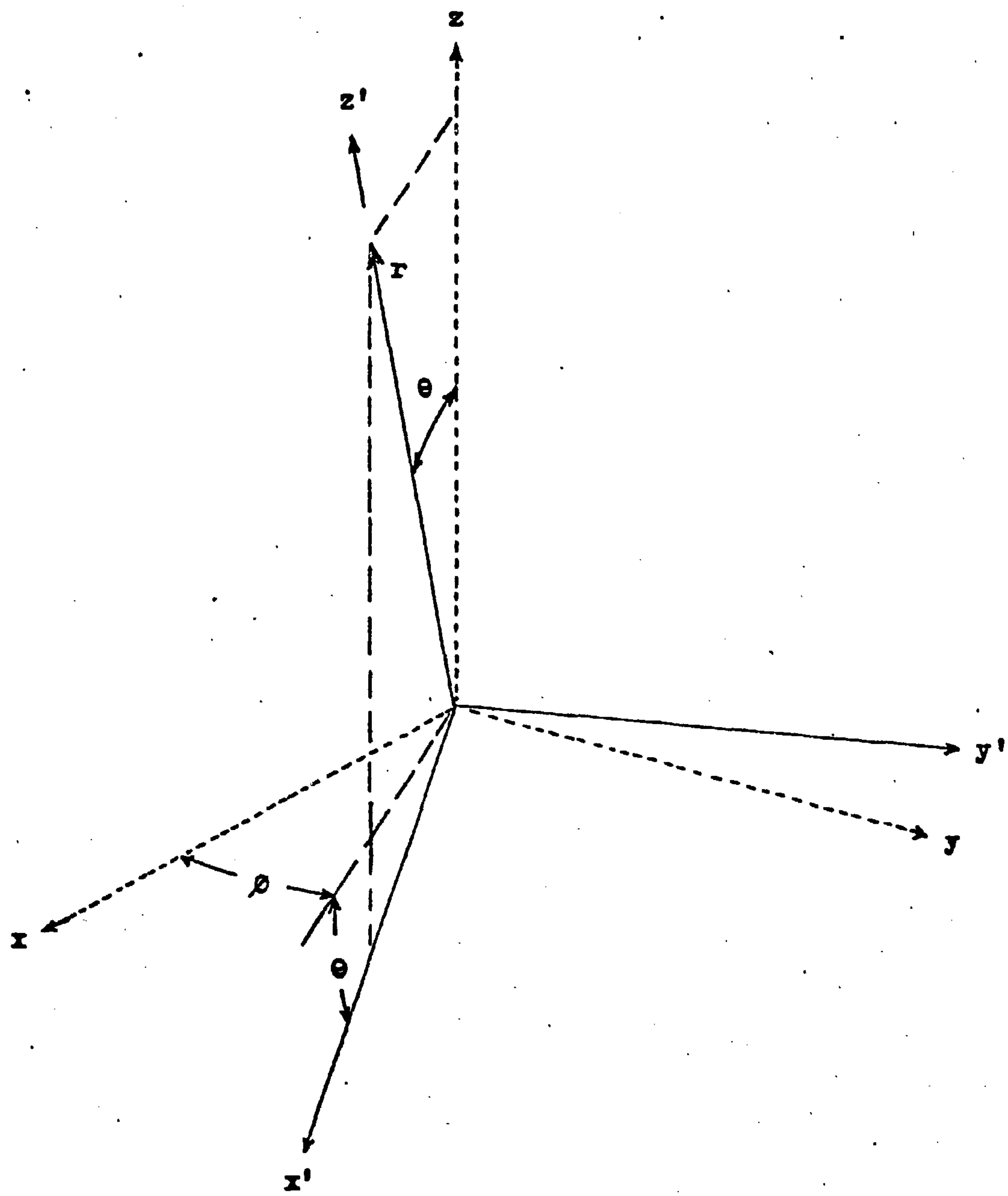

Figure 2. The relationship between the (x',y',z')-axes and the (x,y,z)-axes.

coordinates which is perpendicular to $\underset{\sim}{r}$ will then intersect the (x,y)-plane in a straight line which contains the y'-axis. The direction of this axis can then be obtained, as shown in Fig. 2, through the application of the right-hand rule. If $\underset{\sim}{a}_z$ is a unit vector in the direction of the z-axis, while $\underset{\sim}{r}/r$ is a unit vector in the direction of the z'-axis, then the unit vector in the y'-direction is

$$\underset{\sim}{a}_{y'} = \frac{\underset{\sim}{a}_z \times \underset{\sim}{r}}{r \sin\theta} . \qquad (2.1)$$

If the vector $\underset{\sim}{r}$ happens to be parallel to the z-axis, so that $\sin\theta$ is equal to zero and the vector product $\underset{\sim}{a}_z \times \underset{\sim}{r}$ is also equal to zero, then the direction of the y'-axis becomes indeterminate. Since any value of ϕ can be used in specifying the direction of $\underset{\sim}{r}$, when θ is equal to zero or to π, it is apparent from Fig. 2 that the y'-axis can be directed in any direction within the (x,y)-plane.

It is essential that provision be made for this indeterminacy in the orientation of the y'-axis (and the corresponding indeterminacy in the orientation of the x'-axis) when the angle ϕ' is defined, to accompany the angle θ'.

What is desired is that the definition of ϕ' should permit the specification of the direction of the vector $\underset{\sim}{r}_1$, in terms of the direction of the vector $\underset{\sim}{r}$ and the direction angles θ' and ϕ'. In the ordinary situation, in which $\underset{\sim}{r}$ is not parallel or antiparallel to the z-direction, the y'-axis is uniquely defined and no problem arises. However, when $\underset{\sim}{r}$ lies in the positive or negative z-direction, the angle ϕ is not well defined and it becomes difficult to give the azimuth angle ϕ_1 for the vector $\underset{\sim}{r}_1$, in terms of the ill-defined azimuth angle ϕ of the vector $\underset{\sim}{r}$.

The solution of this problem of azimuth indeterminacy is shown in Fig. 3. With respect to the (x',y',z')-axes, the direction of the vector $\underset{\sim}{r}_1$ is specified by the polar angle θ' and by an azimuth angle which is given by the expression $(\phi' - \phi\cos\theta)$. When $\underset{\sim}{r}$ is not parallel (or antiparallel) to the z-axis, then the use of $(\phi' - \phi\cos\theta)$ instead of ϕ' represents a simple displacement of the azimuth-angle coordinate. However, in the exceptional cases, where $\underset{\sim}{r}$ is parallel (or antiparallel) to the z-axis, the choice of $(\phi' - \phi\cos\theta)$ provides an unequivocal specification of the azimuth angle ϕ_1, even when ϕ itself, and ϕ' accordingly, are both indeterminate.

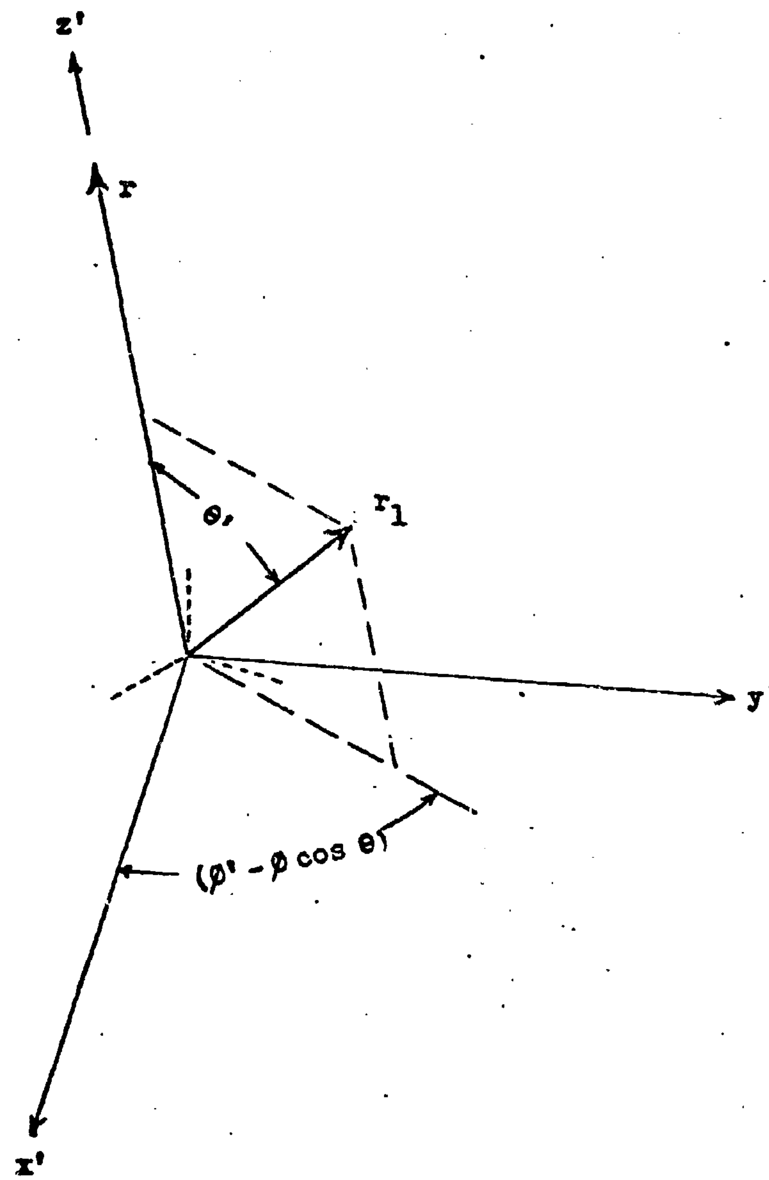

Figure 3. The relationship between the vector $\underset{\sim}{r}_1$ and the (x',y',z')-axes.

3. UNIT VECTORS

In terms of the rectangular set of unit vectors, $\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{k}$, directed along the (x,y,z)-axes, there are three polar-coordinate unit vectors associated with the vector $\underset{\sim}{r}$; these are:

$$
\begin{aligned}
\underset{\sim}{a}_r &= \underset{\sim}{i}\sin\theta\cos\phi + \underset{\sim}{j}\sin\theta\sin\phi + \underset{\sim}{k}\cos\theta , \\
\underset{\sim}{a}_\theta &= \underset{\sim}{i}\cos\theta\cos\phi + \underset{\sim}{j}\cos\theta\sin\phi - \underset{\sim}{k}\sin\theta , \\
\underset{\sim}{a}_\phi &= -\underset{\sim}{i}\sin\phi + \underset{\sim}{j}\cos\phi .
\end{aligned}
\tag{3.1}
$$

A similar set of unit vectors is associated with the vector $\underset{\sim}{r}_1$:

$$
\begin{aligned}
\underset{\sim}{a}_{1r} &= \underset{\sim}{i}\sin\theta_1\cos\phi_1 + \underset{\sim}{j}\sin\theta_1\sin\phi_1 + \underset{\sim}{k}\cos\theta_1 , \\
\underset{\sim}{a}_{1\theta} &= \underset{\sim}{i}\cos\theta_1\cos\phi_1 + \underset{\sim}{j}\cos\theta_1\sin\phi_1 - \underset{\sim}{k}\sin\theta_1 , \\
\underset{\sim}{a}_{1\phi} &= -\underset{\sim}{i}\sin\phi_1 + \underset{\sim}{j}\cos\phi_1 .
\end{aligned}
\tag{3.2}
$$

The primed axes in Fig. 2 have been chosen to lie in the directions of the unit vectors (3.1), so that the primed unit vectors, $\underset{\sim}{i}'$, $\underset{\sim}{j}'$, $\underset{\sim}{k}'$, are given by

$$
\underset{\sim}{i}' = \underset{\sim}{a}_\theta , \quad \underset{\sim}{j}' = \underset{\sim}{a}_\phi , \quad \underset{\sim}{k}' = \underset{\sim}{a}_r . \tag{3.3}
$$

The unit vector which is parallel to the vector $\underset{\sim}{r}_1$ can be written as $\underset{\sim}{a}_{1r}$, in the form given in (3.2), but it can also be expressed in terms of the primed axes:

$$\underset{\sim}{a}_{1r} = \underset{\sim}{i}' \sin\theta' \cos(\phi' - \phi\cos\theta) + \underset{\sim}{j}' \sin\theta' \sin(\phi' - \phi\cos\theta) + \underset{\sim}{k}' \cos\theta' . \qquad (3.4)$$

Equations (3.1-4) yield the trigonometric transformations

$$\cos\theta_1 = \cos\theta\cos\theta' - \sin\theta\sin\theta' \cos(\phi' - \phi\cos\theta) , \qquad (3.5a)$$

$$\sin\theta_1 \cos(\phi_1 - \phi) = \sin\theta\cos\theta' + \cos\theta\sin\theta' \cos(\phi' - \phi\cos\theta) , \qquad (3.5b)$$

$$\sin\theta_1 \sin(\phi_1 - \phi) = \sin\theta' \sin(\phi' - \phi\cos\theta) , \qquad (3.5c)$$

and the inverse transformations

$$\cos\theta' = \cos\theta_1 \cos\theta + \sin\theta_1 \sin\theta \cos(\phi_1 - \phi) , \qquad (3.6a)$$

$$\sin\theta' \cos(\phi' - \phi\cos\theta) = -\cos\theta_1 \sin\theta + \sin\theta_1 \cos\theta \cos(\phi_1 - \phi) , \qquad (3.6b)$$

$$\sin\theta' \sin(\phi' - \phi\cos\theta) = \sin\theta_1 \sin(\phi_1 - \phi) . \qquad (3.6c)$$

4. ADDITION THEOREM

The addition formula (1.1) is based on (3.6a), but a similar formula can readily be written down, based on (3.5a):

$$P_n(\cos\theta_1) = P_n(\cos\theta)\,P_n(\cos\theta')$$
$$+\,2\sum_{\mu=1}^{\mu=n}\frac{(n-\mu)!}{(n+\mu)!}\,P_n^{\mu}(\cos\theta)\,P_n^{\mu}(\cos\theta')\,(-1)^{\mu}\cos\left[\mu(\phi'-\phi\cos\theta)\right]. \quad (4.1)$$

It is this formula which will be generalized into an addition theorem applicable to any spherical harmonic, $X_n^m(\theta_1,\phi_1)$, not just to the Legendre polynomial $P_n(\cos\theta_1)$.

In the notation of Morse and Feshbach[2], a general spherical harmonic can be defined by

$$X_n^m(\theta_1,\phi_1) = \exp(im\phi_1)\,P_n^{|m|}(\cos\theta_1) \;, \quad (4.2)$$

where

$$P_n^{|m|}(\cos\theta_1) = (\sin\theta_1)^{|m|}\frac{d^{|m|}}{d(\cos\theta_1)^{|m|}}P_n(\cos\theta_1) \,. \quad (4.3)$$

It will be assumed that θ, ϕ, θ', ϕ' are independent variables, while θ_1 and ϕ_1 are the dependent functions given by (3.5). Two operators, $D^{+}_{\theta,\phi}$ and $D^{-}_{\theta,\phi}$, will

be defined by

$$D^{\pm}_{\theta,\phi} = \left[\frac{-1}{\sin\theta}\frac{\partial}{\partial\theta} - \frac{\phi}{\cos\theta}\frac{\partial}{\partial\phi} \mp \frac{i}{\sin^2\theta\cos\theta}\frac{\partial}{\partial\phi}\right]. \qquad (4.4)$$

In terms of these operators, the spherical harmonic (4.2) can be written as

$$X_n^m(\theta_1,\phi_1) = \exp(im\phi)\sin^{|m|}\theta\,(D_{\theta,\phi})^m\,P_n(\cos\theta_1), \qquad (4.5)$$

in which the operator expression $(D_{\theta,\phi})^m$ has the meaning $(D^+_{\theta,\phi})^{|m|}$ when m is positive, $(D^-_{\theta,\phi})^{|m|}$ when m is negative. The proof of (4.5) is based on the easily verified results,

$$D^+_{\theta,\phi}\cos\theta_1 = \frac{\exp(i\phi_1)\sin\theta_1}{\exp(i\phi)\sin\theta}, \qquad (4.6a)$$

$$(D^+_{\theta,\phi})^2\cos\theta_1 = 0, \qquad (4.6b)$$

and the complex conjugate equations involving $D^-_{\theta,\phi}$. It is apparent also that

$$X_n^m(\theta,\phi) = \exp(im\phi)\sin^{|m|}\theta\,(D_{\theta,\phi})^m\,P_n(\cos\theta). \qquad (4.7)$$

When $P_n(\cos\theta_1)$ on the right-hand side of (4.5) is replaced by the series expression (4.1), the result is the generalized addition theorem:

$$X_n^m(\theta_1, \phi_1) = X_n^m(\theta, \phi)\; P_n(\cos\theta') + 2 \sum_{\mu=1}^{\mu=n} \frac{(n-\mu)!}{(n+\mu)!} (-1)^{\mu}\; P_n^{\mu}(\cos\theta')$$

$$\cdot \exp(im\phi)\; \sin^{|m|}\theta\; (D_{\theta,\phi})^m \left\{ P_n^{\mu}(\cos\theta) \cos\left[\mu(\phi' - \phi\cos\theta)\right]\right\}. \quad (4.8)$$

Equation (4.8), together with the transformation equations (3.5), constitutes the desired generalization of (1.1) and (1.2). The use of the azimuth-angle expressions $(\phi_1 - \phi)$ and $(\phi' - \phi\cos\theta)$ is a necessary complication, without which the important equation (4.5) could not be established. However, once a choice of (n,m) has been made, and the operations in (4.8) carried out, the angle $(\phi' - \phi\cos\theta)$ can be replaced by a re-defined ϕ', here and in Fig. 3, if this is desired in a particular application of the formula. The replacement cannot be introduced until after the operations $(D_{\theta,\phi})^m$ have been completed.

5. INTEGRAL THEOREM

The addition theorem (4.8) leads directly to an integral theorem. Eq. (4.8) expresses the spherical harmonic $X_n^m(\theta_1, \phi_1)$ as a function of four independent variables, $\theta, \phi, \theta', \phi'$. The dependence upon ϕ' is particularly simple, as can be seen when (4.8) is written in the form

$$X_n^m(\theta_1, \phi_1) = X_n^m(\theta, \phi) P_n(\cos\theta') + \sum_{\mu=1}^{\mu=n} \left[A_c \cos(\mu\phi') + A_s \sin(\mu\phi')\right] , \tag{5.1}$$

where the coefficients A_c and A_s depend upon the variables θ, ϕ, θ' and the parameters n, m, μ, but not upon ϕ'.

Equation (5.1) can now be integrated with respect to ϕ', with the other three variables held constant. Each term in the summation over μ integrates to zero, leaving only:

$$\int_{\phi'=0}^{\phi'=2\pi} X_n^m(\theta_1, \phi_1)\, d\phi' = 2\pi X_n^m(\theta, \phi) P_n(\cos\theta') . \tag{5.2}$$

Other related integral theorems are given in a separate article[3].

REFERENCES

1. P. M. Morse and H. Feshbach, <u>Methods of Theoretical Physics</u> (McGraw-Hill, 1953), page 1274.

2. Reference 1, page 1898.

3. R. E. Clapp and H. T. Li, 'Six Integral Theorems for Vector Spherical Harmonics,' Part III of this Final Report.

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Part III. Six Integral Theorems for Vector Spherical Harmonics . . . Roger E. Clapp and H. Tung Li

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ABSTRACT

With the aid of a generalized addition theorem for spherical harmonics, previously obtained, six integral theorems for vector spherical harmonics are proved. A source-point direction, (θ_1,ϕ_1), is first expressed in terms of a field-point direction, (θ,ϕ), and a polar-coordinate angle-pair, (θ',ϕ'), which has as its polar axis the field-point direction. For a particular choice of n and m, all the components of the vector spherical harmonics for the source, expressed in terms of (θ_1,ϕ_1), are integrated over the relative azimuth angle, ϕ', while the field-point direction, (θ,ϕ), and the relative polar angle, θ', are held fixed. The result in each case is a spherical harmonic or vector spherical harmonic of the field-point direction, with the same n and m but now depending on (θ,ϕ) instead of (θ_1,ϕ_1), multiplied by an explicit function of the relative polar angle, θ'.

1. INTRODUCTION

In the preceding article[1], a generalized addition theorem for spherical harmonics was established. This addition theorem led directly to the proof of an integral theorem for spherical harmonics. In the integration, a vector $\underset{\sim}{r}_1$ is swung about a fixed vector $\underset{\sim}{r}$, with a constant angle θ' separating the two vectors. A suitably defined azimuth angle, ϕ', locates the azimuth of $\underset{\sim}{r}_1$ in its motion about $\underset{\sim}{r}$. This integration, over a full circuit, gave the result:

$$\int_{\phi'=0}^{\phi'=2\pi} X_n^m(\theta_1,\phi_1)\, d\phi' = 2\pi\, X_n^m(\theta,\phi)\, P_n(\cos\theta'), \qquad (1.1)$$

where X_n^m is a general spherical harmonic, and P_n is a Legendre polynomial.

In this article the result (1.1) will be generalized to apply to problems which arise in the use and application of vector spherical harmonics, vector functions which are closely related to the scalar spherical harmonics X_n^m. Because there are three independent vector spherical harmonics for each choice of (n,m), and each of these has three vector components, the generalization of (1.1) leads

to nine independent scalar equations. However, these nine scalar equations group naturally into three radial-component equations, each of scalar form, and three transverse-component equations, each of which combines two scalar equations into a vector equation with only two independent components rather than three.

There are thus only six separate integral theorems to be considered, rather than nine. This article gives the proof of each of these six integral theorems.

2. VECTOR SPHERICAL HARMONICS

The notation for the vector spherical harmonics, to be used in this article, follows that used by Morse and Feshbach[2]. The scalar spherical harmonics are defined by

$$X_n^m(\theta,\phi) = \exp(im\phi)\, P_n^{|m|}(\cos\theta)\,, \tag{2.1}$$

where

$$P_n^{|m|}(\cos\theta) = (\sin\theta)^{|m|} \frac{d^{|m|}}{d(\cos\theta)^{|m|}} P_n(\cos\theta)\,. \tag{2.2}$$

The vector spherical harmonics separate into three sets:

$$\underset{\sim}{P}{}_n^m(\theta,\phi) = \underset{\sim}{a}_r\, X_n^m(\theta,\phi)\,, \tag{2.3}$$

$$\underset{\sim}{B}{}_n^m(\theta,\phi) = \frac{\sqrt{n(n+1)}}{(2n+1)\sin\theta}\left\{\underset{\sim}{a}_\theta\left[\frac{(n-|m|+1)}{(n+1)} X_{n+1}^m - \frac{(n+|m|)}{n} X_{n-1}^m\right] + \underset{\sim}{a}_\phi \frac{i\,m\,(2n+1)}{n(n+1)} X_n^m\right\}, \tag{2.4}$$

$$\underset{\sim}{C}{}_n^m(\theta,\phi) = \frac{\sqrt{n(n+1)}}{(2n+1)\sin\theta}\left\{\underset{\sim}{a}_\theta \frac{i\,m\,(2n+1)}{n(n+1)} X_n^m - \underset{\sim}{a}_\phi\left[\frac{(n-|m|+1)}{(n+1)} X_{n+1}^m - \frac{(n+|m|)}{n} X_{n-1}^m\right]\right\}. \tag{2.5}$$

In (2.4) and (2.5) the arguments of the scalar harmonics are (θ,ϕ) in each case. Explicit forms for the unit vectors $\underset{\sim}{a}_r$, $\underset{\sim}{a}_\theta$, and $\underset{\sim}{a}_\phi$ are given in Eqs. (3.1) of Reference 1.

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$$\underset{\sim}{C}_n^m(\theta,\phi) = \frac{\sqrt{n(n+1)}}{(2n+1)\sin\theta}\left\{\underset{\sim}{a}_\theta \frac{im(2n+1)}{n(n+1)} X_n^m - \underset{\sim}{a}_\phi\left[\frac{(n-|m|+1)}{(n+1)} X_{n+1}^m - \frac{(n+|m|)}{n} X_{n-1}^m\right]\right\}. \tag{2.5}$$

In (2.4) and (2.5) the arguments of the scalar harmonics are (θ,ϕ) in each case. Explicit forms for the unit vectors $\underset{\sim}{a}_r$, $\underset{\sim}{a}_\theta$, and $\underset{\sim}{a}_\phi$ are given in Eqs. (3.1) of Reference 1.

3. THEOREM I

The six integral theorems involve integrations over ϕ' of components of the vector spherical harmonics which are expressed as functions of (θ_1, ϕ_1). For example, the vector spherical harmonic $\underset{\sim}{P}_n^m(\theta_1, \phi_1)$ has the form

$$\underset{\sim}{P}_n^m(\theta_1, \phi_1) = \underset{\sim}{a}_{1r}\, X_n^m(\theta_1, \phi_1), \qquad (3.1)$$

obtained from (2.3), where the unit vector $\underset{\sim}{a}_{1r}$, and the orthogonal unit vectors $\underset{\sim}{a}_{1\theta}$ and $\underset{\sim}{a}_{1\phi}$, have been given explicitly in Eqs. (3.2) of Reference 1. The geometrical configuration of the vectors $\underset{\sim}{r}$ and $\underset{\sim}{r}_1$ is shown in Figs. 1-3 of Ref. 1, while the trigonometric transformations are given in Eqs. (3.5) and (3.6) of Ref. 1.

In the integration over ϕ', the vector $\underset{\sim}{r}$ is held fixed, while the vector $\underset{\sim}{r}_1$ is swung in an arc with the angle θ' held constant. The path of integration is shown here in Fig. 1. Of the four independent angular variables, $\theta, \phi, \theta', \phi'$, only ϕ' varies, but both of the angles θ_1 and ϕ_1 will vary during the integration. These latter are treated as dependent variables, through the use of Eqs. (3.5) of Ref. 1.

The direction of the vector function (3.1) changes during the integration over ϕ', since $\underset{\sim}{a}_{1r}$ is a function of ϕ'.

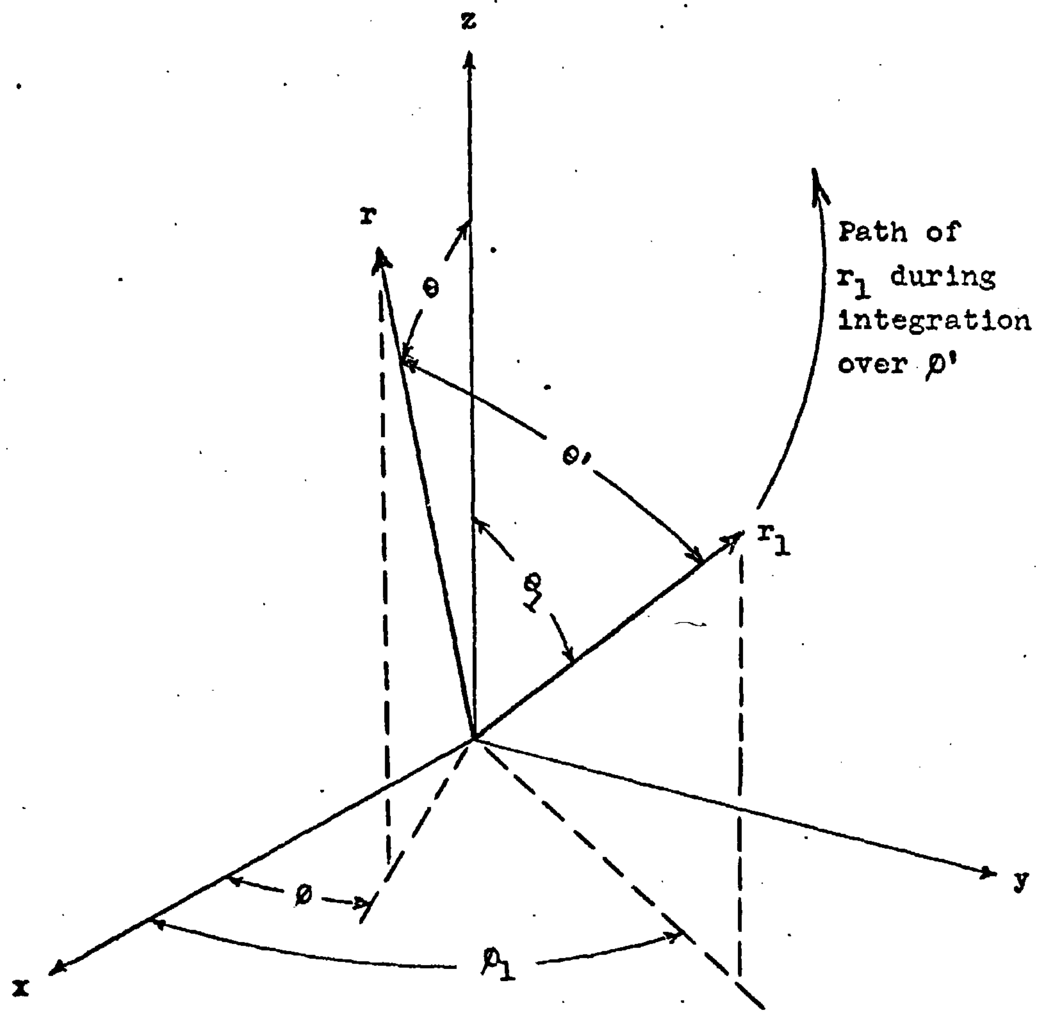

Figure 1. The relationship between the source-point vector $\underset{\sim}{r}_1$ and the field-point vector $\underset{\sim}{r}$. During the integration over ϕ' the vector $\underset{\sim}{r}_1$ is swung about $\underset{\sim}{r}$ with the relative polar angle θ' held constant.

The vector function (3.1) will therefore be resolved into its components before the integration, and the simplest component to be considered is the component which lies in the z'-direction, the direction of the unit vector $\underset{\sim}{a}_r$. Accordingly, the projection of (3.1) onto $\underset{\sim}{a}_r$,

$$\underset{\sim}{a}_r \cdot \underset{\sim}{P}_n^m(\theta_1,\phi_1) = \cos\theta' \, X_n^m(\theta_1,\phi_1) , \qquad (3.2)$$

will be integrated over ϕ'.

When the generalized addition theorem,

$$X_n^m(\theta_1,\phi_1) = X_n^m(\theta,\phi)\, P_n(\cos\theta') + \sum_{\mu=1}^{\mu=n} \left[A_c \cos(\mu\phi') + A_s \sin(\mu\phi') \right] , \qquad (3.3)$$

which was given as Eq. (5.1) of Ref. 1, is substituted in the right-hand side of (3.2), and the resulting expression integrated term by term over the angle ϕ', with θ,ϕ,θ' held constant, each term in the summation over μ integrates to zero (since A_c and A_s do not depend upon ϕ'), and the result is the first of the six integral theorems:

$$\int_{\phi'=0}^{\phi'=2\pi} \underset{\sim}{a}_r \cdot \underset{\sim}{P}_n^m(\theta_1,\phi_1)\, d\phi' = 2\pi X_n^m(\theta,\phi) \cos\theta' \, P_n(\cos\theta') . \qquad (I)$$

4. THEOREMS II AND III

Two more integral theorems are obtained when the addition theorem (3.3) is differentiated with respect to θ' and with respect to ϕ'. It can first be verified, from Eqs. (3.1), (3.2), and (3.5) of Ref. 1, that:

$$\frac{\partial\theta_1}{\partial\theta'} = -\frac{1}{\sin\theta'}(\underset{\sim}{a}_r \cdot \underset{\sim}{a}_{1\theta}) , \qquad (4.1a)$$

$$\frac{\partial\phi_1}{\partial\theta'} = \frac{-1}{\sin\theta_1 \sin\theta'}(\underset{\sim}{a}_r \cdot \underset{\sim}{a}_{1\phi}) , \qquad (4.1b)$$

$$\frac{\partial\theta_1}{\partial\phi'} = (\underset{\sim}{a}_r \cdot \underset{\sim}{a}_{1\phi}) , \qquad (4.2a)$$

$$\frac{\partial\phi_1}{\partial\phi'} = \frac{-1}{\sin\theta_1}(\underset{\sim}{a}_r \cdot \underset{\sim}{a}_{1\theta}) . \qquad (4.2b)$$

From the properties of the spherical harmonics (2.1) it can also be verified that

$$\frac{\partial}{\partial\theta_1} X_n^m(\theta_1,\phi_1) = \frac{n\,(n+1)}{(2n+1)\sin\theta_1}\left[\frac{(n-|m|+1)}{(n+1)} X_{n+1}^m(\theta_1,\phi_1) - \frac{(n+|m|)}{n} X_{n-1}^m(\theta_1,\phi_1)\right] , \qquad (4.3$$

$$\frac{\partial}{\partial\phi_1} X_n^m(\theta_1,\phi_1) = i\,m\,X_n^m(\theta_1,\phi_1) . \qquad (4.3$$

From (4.1-3) and from the definitions of $\underset{\sim}{B}_n^m$ and $\underset{\sim}{C}_n^m$ in (2.4-5) it can be seen that

$$\frac{\partial}{\partial\theta'} X_n^m(\theta_1,\phi_1) = -\frac{\sqrt{n(n+1)}}{\sin\theta'}\, \underset{\sim}{a}_r\cdot \underset{\sim}{B}_n^m(\theta_1,\phi_1) \,, \qquad (4.4)$$

$$\frac{\partial}{\partial\phi'} X_n^m(\theta_1,\phi_1) = -\sqrt{n(n+1)}\, \underset{\sim}{a}_r\cdot \underset{\sim}{C}_n^m(\theta_1,\phi_1) \,. \qquad (4.5)$$

Differentiation of (3.3) with respect to θ' now gives

$$\underset{\sim}{a}_r\cdot \underset{\sim}{B}_n^m(\theta_1,\phi_1) = \frac{1}{\sqrt{n(n+1)}}\left\{X_n^m(\theta,\phi)\,\sin^2\theta'\,\frac{d}{d(\cos\theta')}P_n(\cos\theta') - \sin\theta'\sum_{\mu=1}^{n}\left[\frac{\partial A_c}{\partial\theta'}\cos(\mu\phi') + \frac{\partial A_s}{\partial\theta'}\sin(\mu\phi')\right]\right\}, \qquad (4.6)$$

while differentiation with respect to ϕ' gives

$$\underset{\sim}{a}_r\cdot \underset{\sim}{C}_n^m(\theta_1,\phi_1) = \frac{1}{\sqrt{n(n+1)}}\sum_{\mu=1}^{n}\left[\mu A_c\sin(\mu\phi') - \mu A_s\cos(\mu\phi')\right]. \qquad (4.7)$$

When the explicit forms for A_c and A_s, obtainable from Eq. (4.8) of Ref. 1, are introduced, then (4.6) and (4.7) become addition theorems in their own right, for use with the vector spherical harmonics $\underset{\sim}{B}_n^m$ and $\underset{\sim}{C}_n^m$.

Equations (4.6) and (4.7) can now be integrated with respect to ϕ', to give the second and third integral theorems:

$$\int_{\phi'=0}^{\phi'=2\pi} \underset{\sim}{a}_r \cdot \underset{\sim}{B}_n^m(\theta_1,\phi_1)\ d\phi' = \frac{2\pi}{\sqrt{n(n+1)}} X_n^m(\theta,\phi)\ \sin^2\theta' \frac{d}{d(\cos\theta')} P_n(\cos\theta' \tag{II}$$

$$\int_{\phi'=0}^{\phi'=2\pi} \underset{\sim}{a}_r \cdot \underset{\sim}{C}_n^m(\theta_1,\phi_1)\ d\phi' = 0\ . \tag{III}$$

5. RECURSION FORMULAS

The familiar recursion formula for spherical harmonics[3],

$$\cos\theta\, X_n^m(\theta,\phi) = \frac{(n-|m|+1)}{(2n+1)} X_{n+1}^m(\theta,\phi) + \frac{(n+|m|)}{(2n+1)} X_{n-1}^m(\theta,\phi)\,, \qquad (5.1)$$

leads directly to a similar formula for the vector spherical harmonics $\underset{\sim}{P}_n^m(\theta,\phi)$:

$$\cos\theta\, \underset{\sim}{P}_n^m(\theta,\phi) = \frac{(n-|m|+1)}{(2n+1)} \underset{\sim}{P}_{n+1}^m(\theta,\phi) + \frac{(n+|m|)}{(2n+1)} \underset{\sim}{P}_{n-1}^m(\theta,\phi)\,. \qquad (5.2)$$

Corresponding recursion formulas can be established for the vector spherical harmonics $\underset{\sim}{B}_n^m(\theta,\phi)$ and $\underset{\sim}{C}_n^m(\theta,\phi)$, through (5.1) as applied to (2.4) and (2.5), but there is some cross-coupling of the two symmetries:

$$\cos\theta\, \underset{\sim}{B}_n^m(\theta,\phi) = \frac{(n-|m|+1)\sqrt{n(n+2)}}{(2n+1)\;(n+1)} \underset{\sim}{B}_{n+1}^m(\theta,\phi) - \frac{i\,m}{n(n+1)} \underset{\sim}{C}_n^m(\theta,\phi)$$
$$+ \frac{(n+|m|)\sqrt{(n-1)(n+1)}}{(2n+1)\;n} \underset{\sim}{B}_{n-1}^m(\theta,\phi)\,, \qquad (5.3)$$

$$\cos\theta\, \underset{\sim}{C}_n^m(\theta,\phi) = \frac{(n-|m|+1)\sqrt{n(n+2)}}{(2n+1)\;(n+1)} \underset{\sim}{C}_{n+1}^m(\theta,\phi) + \frac{i\,m}{n(n+1)} \underset{\sim}{B}_n^m(\theta,\phi)$$
$$+ \frac{(n+|m|)\sqrt{(n-1)(n+1)}}{(2n+1)\;n} \underset{\sim}{C}_{n-1}^m(\theta,\phi)\,. \qquad (5.4)$$

6. THEOREM IV

From the equations in Section 3 of Reference 1, it can be shown that

$$(\underset{\sim}{a}_\theta \cdot \underset{\sim}{a}_{1r}) = \sin\theta' \cos(\phi' - \phi \cos\theta) = \frac{1}{\sin\theta} (\cos\theta \cos\theta' - \cos\theta_1), \quad (6.1$$

$$(\underset{\sim}{a}_\phi \cdot \underset{\sim}{a}_{1r}) = \sin\theta' \sin(\phi' - \phi \cos\theta) = \frac{1}{\sin\theta} \frac{\partial}{\partial\phi'}(\cos\theta_1) \ . \quad (6.2)$$

From (2.3), (5.1), and (6.1) it is apparent that

$$\underset{\sim}{a}_\theta \cdot \underset{\sim}{P}_n^m(\theta_1,\phi_1) = \frac{-1}{\sin\theta}\left[\frac{(n-|m|+1)}{(2n+1)} X_{n+1}^m(\theta_1,\phi_1) - \cos\theta\cos\theta' X_n^m(\theta_1,\phi_1) + \frac{(n+|m|)}{(2n+1)} X_{n-1}^m(\theta_1,\phi_1)\right] . \quad (6.3)$$

Each of the scalar spherical harmonics on the right-hand side of (6.3) can be replaced by an expansion of the form (3.3). Integration over ϕ' then gives

$$\int_{\phi'=0}^{\phi'=2\pi} \underset{\sim}{a}_\theta \cdot \underset{\sim}{P}_n^m(\theta_1,\phi_1)\, d\phi' = \frac{-2\pi}{\sin\theta}\left\{\frac{(n-|m|+1)}{(2n+1)} X_{n+1}^m(\theta,\phi)\, P_{n+1}(\cos\theta') - \left[\cos\theta\, X_n^m(\theta,\phi)\right]\left[\cos\theta'\, P_n(\cos\theta')\right] + \frac{(n+|m|)}{(2n+1)} X_{n-1}^m(\theta,\phi)\, P_{n-1}(\cos\theta')\right\} \quad (6.4)$$

Eq. (6.4) can be simplified with the aid of (5.1), and with the use of (4.3a) in the form it takes with $m = 0$. The result is the first half of the fourth integral theorem:

$$\int_{\phi'=0}^{\phi'=2\pi} \underset{\sim}{a}_\theta \cdot \underset{\sim}{P}_n^m(\theta_1, \phi_1)\, d\phi' = \frac{2\pi}{(2n+1)\sin\theta}\left[\frac{(n-|m|+1)}{(n+1)} X_{n+1}^m(\theta,\phi) - \frac{(n+|m|)}{n} X_{n-1}^m(\theta,\phi)\right] \sin^2\theta' \frac{d}{d(\cos\theta')} P_n(\cos\theta') \tag{IVa}$$

From (2.3), (6.2), and (4.5), the relationship

$$\underset{\sim}{a}_\phi \cdot \underset{\sim}{P}_n^m(\theta_1, \phi_1) = \frac{1}{\sin\theta}\frac{\partial}{\partial\phi'}\left[\cos\theta_1 X_n^m(\theta_1,\phi_1)\right] + \frac{\sqrt{n(n+1)}}{\sin\theta}\underset{\sim}{a}_r \cdot \left[\cos\theta_1 \underset{\sim}{C}_n^m(\theta_1,\phi_1)\right] \tag{6.5}$$

can be established, and the recursion formulas (5.1) and (5.4) can then be used to replace the bracketed expressions by linear combinations of scalar or vector harmonics. The integration over ϕ' can then be carried out without difficulty, by methods used in earlier theorems, with the result:

$$\int_{\phi'=0}^{\phi'=2\pi} \underset{\sim}{a}_\phi \cdot \underset{\sim}{P}_n^m(\theta_1, \phi_1)\, d\phi' = \frac{2\pi i\, m}{n(n+1)\sin\theta} X_n^m(\theta,\phi) \sin^2\theta' \frac{d}{d(\cos\theta')} P_n(\cos\theta') . \tag{IVb}$$

Theorems (IVa) and (IVb) deal with the transverse components of $\underset{\sim}{P}_n^m(\theta_1,\phi_1)$, that is, the components that are perpendicular to the unit vector $\underset{\sim}{a}_r$. A vector combination of just these transverse components is equivalent to the subtraction of the longitudinal component from the original vector:

$$\underset{\sim}{P}_n^m(\theta_1,\phi_1) - \underset{\sim}{a}_r\underset{\sim}{a}_r\cdot\underset{\sim}{P}_n^m(\theta_1,\phi_1) = \underset{\sim}{a}_\theta\,\underset{\sim}{a}_\theta\cdot\underset{\sim}{P}_n^m(\theta_1,\phi_1) + \underset{\sim}{a}_\phi\,\underset{\sim}{a}_\phi\cdot\underset{\sim}{P}_n^m(\theta_1,\phi_1) \tag{6.6}$$

When this vector combination of (IVa) and (IVb) is made, it is found to have the grouping of terms that appears in (2.4). The result is:

$$\int_{\phi'=0}^{\phi'=2\pi}\left[\underset{\sim}{P}_n^m(\theta_1,\phi_1) - \underset{\sim}{a}_r\underset{\sim}{a}_r\cdot\underset{\sim}{P}_n^m(\theta_1,\phi_1)\right]d\phi'$$

$$= \frac{2\pi}{\sqrt{n(n+1)}}\,\underset{\sim}{B}_n^m(\theta,\phi)\,\sin^2\theta'\,\frac{d}{d(\cos\theta')}\,P_n(\cos\theta')\,. \tag{IV}$$

7. THEOREM V

The fifth integral theorem deals with the transverse part of the vector spherical harmonic $\mathbf{B}_n^m(\theta_1, \phi_1)$. In analogy with (6.1-2), four scalar products will be needed:

$$(\mathbf{a}_\theta \cdot \mathbf{a}_{1\theta}) = \frac{\sin\theta_1}{\sin\theta} - \frac{\cos\theta \sin\theta'}{\sin\theta} \frac{\partial\theta_1}{\partial\theta'} , \tag{7.1a}$$

$$(\mathbf{a}_\theta \cdot \mathbf{a}_{1\phi}) = - \frac{\sin\theta_1 \cos\theta \sin\theta'}{\sin\theta} \frac{\partial\phi_1}{\partial\theta'} , \tag{7.1b}$$

$$(\mathbf{a}_\phi \cdot \mathbf{a}_{1\theta}) = - \frac{\cos\theta_1}{\sin\theta} \frac{\partial\theta_1}{\partial\phi'} , \tag{7.1c}$$

$$(\mathbf{a}_\phi \cdot \mathbf{a}_{1\phi}) = \frac{\sin\theta_1 \cos\theta'}{\sin\theta} - \frac{\sin\partial_1 \cos\theta_1}{\sin\theta} \frac{\partial\phi_1}{\partial\phi'} , \tag{7.1d}$$

which can all be derived from the equations in Section 3 of Reference 1.

From (2.4), (4.3), and (7.1) it is found that the transverse portion of $\mathbf{B}_n^m(\theta_1, \phi_1)$ is:

$$\mathbf{a}_\theta\, \mathbf{a}_\theta \cdot \mathbf{B}_n^m(\theta_1, \phi_1) \;+\; \mathbf{a}_\phi\, \mathbf{a}_\phi \cdot \mathbf{B}_n^m(\theta_1, \phi_1)$$

$$= \frac{-1}{\sqrt{n(n+1)}} \left[\mathbf{a}_\theta \, \frac{\cos\theta \sin\theta'}{\sin\theta} \frac{\partial}{\partial\theta'} X_n^m(\theta_1, \phi_1) \right.$$

$$\left. + \; \mathbf{a}_\phi \, \frac{\cos\theta_1}{\sin\theta} \frac{\partial}{\partial\phi'} X_n^m(\theta_1, \phi_1) \right]$$

$$+ \frac{\sqrt{n(n+1)}}{(2n+1)\sin\theta} \, \mathbf{a}_\theta \left[\frac{(n-|m|+1)}{(n+1)} X_{n+1}^m(\theta_1, \phi_1) \;-\; \frac{(n+|m|)}{n} X_{n-1}^m(\theta_1, \phi_1) \right]$$

$$+ \frac{i\, m \cos\theta'}{\sqrt{n(n+1)}\, \sin\theta} \, \mathbf{a}_\phi \, X_n^m(\theta_1, \phi_1) \,. \tag{7.2}$$

In the integration of (7.2) over ϕ', there are five terms to be considered. The first two terms can be transformed with the aid of (4.4), (4.5), and (5.4), then integrated through Theorems II and III. The remaining three terms may be integrated directly, by Theorem I or its scalar equivalent, Eq. (5.2) of Reference 1. What is obtained from these integrations is the fifth integral theorem:

$$\int_{\phi'=0}^{\phi'=2\pi} \left[\mathbf{B}_n^m(\theta_1, \phi_1) \;-\; \mathbf{a}_r\, \mathbf{a}_r \cdot \mathbf{B}_n^m(\theta_1, \phi_1) \right] d\phi'$$

$$= 2\pi \; \mathbf{B}_n^m(\theta, \phi) \left[\cos\theta' \; P_n(\cos\theta') \;+\; \frac{\sin^2\theta'}{n(n+1)} \frac{d}{d(\cos\theta')} P_n(\cos\theta') \right] . \tag{V}$$

8. THEOREM VI

Four additional partial derivatives, analogous to those given in (4.1) and (4.2), can be obtained from Eqs. (3.5) of Reference 1:

$$\frac{\partial\theta_1}{\partial\theta} = \frac{1}{\sin\theta_1}\Big[\sin\theta\cos\theta' + \cos\theta\sin\theta'\cos(\phi'-\phi\cos\theta)$$
$$- \phi\sin^2\theta\sin\theta'\sin(\phi'-\phi\cos\theta)\Big], \qquad (8.1a)$$

$$\frac{\partial\theta_1}{\partial\phi} = \frac{1}{\sin\theta_1}\sin\theta\cos\theta\sin\theta'\sin(\phi'-\phi\cos\theta), \qquad (8.1b)$$

$$\frac{\partial\phi_1}{\partial\theta} = \frac{\sin\theta'}{\sin^2\theta_1}\Big[-\cos\theta\cos\theta'\sin(\phi'-\phi\cos\theta)$$
$$+ \sin\theta\sin\theta'\sin(\phi'-\phi\cos\theta)\cos(\phi'-\phi\cos\theta)$$
$$+ \phi\sin\theta\cos\theta\sin\theta'$$
$$+ \phi\sin^2\theta\cos\theta'\cos(\phi'-\phi\cos\theta)\Big]. \qquad (8.1c)$$

$$\frac{\partial\phi_1}{\partial\phi} = \frac{\sin\theta}{\sin^2\theta_1}\Big[\sin\theta + \cos\theta\sin\theta'\cos\theta'\cos(\phi'-\phi\cos\theta)$$
$$- \sin\theta\sin^2\theta'\cos^2(\phi'-\phi\cos\theta)\Big]. \qquad (8.1d)$$

In terms of these derivatives, (7.1) can be rewritten as

$$(\underset{\sim}{a}_\theta \cdot \underset{\sim}{a}_{1\theta}) = \frac{\sin\theta_1}{\sin\theta} \frac{\partial\phi_1}{\partial\phi}, \tag{8.2a}$$

$$(\underset{\sim}{a}_\theta \cdot \underset{\sim}{a}_{1\phi}) = \frac{-1}{\sin\theta} \frac{\partial\theta_1}{\partial\phi}, \tag{8.2b}$$

$$(\underset{\sim}{a}_\phi \cdot \underset{\sim}{a}_{1\theta}) = -\sin\theta_1 \left[\frac{\partial\phi_1}{\partial\theta} - \phi\tan\theta + \phi\tan\theta \frac{\partial\phi_1}{\partial\phi} \right] \tag{8.2c}$$

$$(\underset{\sim}{a}_\phi \cdot \underset{\sim}{a}_{1\phi}) = \frac{\partial\theta_1}{\partial\theta} + \phi\tan\theta \frac{\partial\theta_1}{\partial\phi}. \tag{8.2d}$$

In analogy with (7.2), the transverse portion of the vector spherical harmonic $\underset{\sim}{C}_n^m(\theta_1,\phi_1)$ can be written in the form:

$$\underset{\sim}{a}_\theta\, \underset{\sim}{a}_\theta \cdot \underset{\sim}{C}_n^m(\theta_1,\phi_1) + \underset{\sim}{a}_\phi\, \underset{\sim}{a}_\phi \cdot \underset{\sim}{C}_n^m(\theta_1,\phi_1)$$

$$= \frac{1}{\sqrt{n(n+1)}} \left\{ \underset{\sim}{a}_\theta \frac{1}{\sin\theta} \frac{\partial}{\partial\phi} X_n^m(\theta_1,\phi_1) - \underset{\sim}{a}_\phi \left[\frac{\partial}{\partial\theta} - i\,m\,\phi\tan\theta + \phi\tan\theta \frac{\partial}{\partial\phi} \right] X_n^m(\theta_1,\phi_1) \right\}. \tag{8.3}$$

When $X_n^m(\theta_1,\phi_1)$ in (8.3) is replaced by the expansion (3.3), and the resulting expression integrated over ϕ', the sixth of the integral theorems is obtained:

$$\int_{\phi'=0}^{\phi'=2\pi} \left[\underset{\sim}{C}_n^m(\theta_1,\phi_1) - \underset{\sim}{a}_r\, \underset{\sim}{a}_r \cdot \underset{\sim}{C}_n^m(\theta_1,\phi_1) \right] d\phi' = 2\pi\, \underset{\sim}{C}_n^m(\theta,\phi)\, P_n(\cos\theta') \, . \tag{VI}$$

9. APPLICATIONS

These six integral theorems are of particular value when a vector field function is to be obtained from a vector source function. An example is the retarded Hertz vector, as obtained from an electrical source-current distribution.[4] The theorems show that if a source function, depending on the angles (θ_1, ϕ_1), has that dependence characterized by a particular choice of n and m, then the integration over the azimuth angle ϕ' will immediately ensure that the field function, in its dependence upon the angles (θ, ϕ), will be characterized by the same choice of n and m, provided that the Green's function relating field to source does not itself depend upon ϕ'. Whatever dependence the Green's function may have upon the angle θ' (the angle between the vector $\mathbf{r}_1$ to the source point and the vector $\mathbf{r}$ to the field point) is here immaterial, and cannot affect the 'mode separation' of the expansion into vector spherical harmonics.

It should be noted, however, that certain of the six theorems introduce coupling between a $\mathbf{P}_n^m$ source function and a $\mathbf{B}_n^m$ field function, or vice versa. That is, the theorems mix the $\mathbf{P}_n^m$ and $\mathbf{B}_n^m$ symmetries, while keeping the $\mathbf{C}_n^m$ symmetry separate from the other two.

The use of these theorems in electromagnetism will be illustrated in a following article[4].

REFERENCES

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2. P. M. Morse and H. Feshbach, Methods of Theoretical Physics (McGraw-Hill, 1953), pages 1898-1899.

3. Reference 2, page 1326.

4. R. E. Clapp, L. Huang, and H. T. Li, 'Vector Spherical Harmonic Expansion in the Time Domain of the Retarded Hertz Vector for a Distributed, Transient Source-Current Configuration,' Part IV of this Final Report.

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Part IV. Vector Spherical Harmonic Expansion in the Time Domain of the Retarded Hertz Vector for a Distributed, Transient Source-Current Configuration . . . .

. . . Roger E. Clapp, Luc Huang, and H. Tung Li

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ABSTRACT

An [illegible] equation giving the retarded Hertz vector, $\underset{\sim}{\Pi}(\underset{\sim}{r},t)$, in terms of a source-current distribution, $\underset{\sim}{J}(\underset{\sim}{r}_1,t_1)$, is derived. Both vector functions are written as expansions in vector spherical harmonics, with the expansion coefficients containing the dependence upon radial distance and time, while the angular dependence is kept within the harmonic functions. After integration over two angles, expressions are obtained giving the expansion coefficients for the Hertz vector, which depend upon (r,t), in terms of the corresponding expansion coefficients for the current, which depend upon (r_1,t_1). The original four-dimensional problem is thus reduced to two dimensions, but with the four-dimensional causality requirements satisfied at each step of the analysis.

1. INTRODUCTION

Although the electromagnetic radiation from a general time-dependent source-current distribution is usually analyzed in the frequency domain, for certain problems the time domain is more appropriate. An example is the electromagnetic radiation from the electrical currents generated in the air by an atmospheric nuclear detonation[1].

In this example the source currents are highly transient, and retardation across the source region plays an important role. Furthermore, part of the current is in the form of relativistic electrons, produced by Compton collisions between gamma-rays from the detonation and electrons from air molecules. The fields radiated by relativistic electrons show forward directivity, which is not easily expressed in the frequency domain but which enters readily into a time-domain formalism.

The analysis is most straightforward if the medium is assumed to be the vacuum. In the example[1] mentioned above, there is a time-dependent conductivity in the source region, but this conductivity is actually formed of electrons and ions whose motion in the local fields can be represented as an addition to the primary source current. Similarly,

dielectric polarization of the air molecules can also be represented mathematically as a secondary increment to the source-current distribution. Thus if the secondary currents are all treated explicitly, the vacuum equations will be adequate for the calculation of the fields generated by all the source currents.

The source-current distribution will be expressed as an expansion in vector spherical harmonics, with each coefficient depending in an arbitrary way upon the radial distance and the time. With the aid of theorems established earlier[2], the angular dependence of the source will be integrated over, leaving the field function for each vector-spherical-harmonic mode expressed as a function of radial distance and time.

The Hertz-vector formalism will be used, because of the relative simplicity of the integrations. In later articles[3,4] the magnetic field and electric field will be given explicitly.

2. HERTZ VECTOR FORMALISM

The electromagnetic fields generated by a time-dependent distribution of charges and currents could be expressed in terms of the vector and scalar potentials, together with an auxiliary condition which limited the charge and current densities to those which satisfied the equation of continuity. It is more convenient, however, to make use of the Hertz vector formalism.[5] The need for an auxiliary condition is avoided through the use of the free-charge polarization vector, $\underset{\approx}{P}$, to represent the source distribution. As will be shown, the integrals can then be reformulated so that the final equation for the Hertz vector expresses the source in terms of the current density, $\underset{\sim}{J}$, without any explicit appearance of the charge density, ρ.

The free-charge polarization vector is defined through the equations

$$\underset{\sim}{J} = \frac{\partial}{\partial t} \underset{\approx}{P} , \qquad \rho = - \underset{\sim}{\nabla} \cdot \underset{\approx}{P} . \tag{2.1}$$

For a transient source, therefore,

$$\underset{\approx}{P} = \int_{t=t_o}^{t=t} \underset{\sim}{J}\, dt \,, \qquad (2.2)$$

where t_o is a time which precedes any current flow. (Any pre-existing electrostatic field can be represented through a constant of integration, $\underset{\approx}{P}_o$, added to the right-hand side of (2.2).) Because of the form of (2.1), the equation of continuity of charge is automatically satisfied.

The Hertz vector, $\underset{\sim}{\Pi}(\underset{\sim}{r},t)$, is defined by

$$\underset{\sim}{\Pi}(\underset{\sim}{r},t) = \frac{1}{4\pi\epsilon_o} \iiint \frac{1}{s} \underset{\approx}{P}(\underset{\sim}{r}_1, t-s/c)\, dV_1 \,, \qquad (2.3)$$

in the MKS units used by Stratton[5]. In (2.3), dV_1 is the volume element at the source point, $\underset{\sim}{r}_1$, and s is the distance from the source point to the field point, $\underset{\sim}{r}$:

$$s = \left| \underset{\sim}{r} - \underset{\sim}{r}_1 \right| . \qquad (2.4)$$

Eq. (2.3) defines the <u>retarded</u> Hertz vector, from which the retarded electric and magnetic field vectors can be obtained, through the equations

$$\underset{\sim}{E} = \underset{\sim}{\nabla}\underset{\sim}{\nabla}\cdot\underset{\sim}{\Pi} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\underset{\sim}{\Pi}, \tag{2.5}$$

$$\underset{\sim}{H} = \epsilon_o \underset{\sim}{\nabla}\times\frac{\partial}{\partial t}\underset{\sim}{\Pi}. \tag{2.6}$$

These field vectors satisfy Maxwell's equations.

An <u>advanced</u> Hertz vector could also be defined, through the use of $\underset{\approx}{P}(\underset{\sim}{r}_1,t+s/c)$ instead of $\underset{\approx}{P}(\underset{\sim}{r}_1,t-s/c)$ in the integrand of (2.3). Eqs. (2.5-6) would then give advanced electromagnetic field vectors. These would also satisfy Maxwell's equations. However, there is no evidence that these advanced fields play any role in macroscopic electromagnetism. Accordingly, only the retarded Hertz vector (2.3) will be considered further in this article.

3. EXPANSION IN VECTOR SPHERICAL HARMONICS

The scalar and vector spherical harmonics have been defined in Section 2 of Reference 2. The vector functions, $\mathbf{J}$, $\mathbf{P}$, and $\boldsymbol{\Pi}$, will be expanded in terms of the vector spherical harmonics, $\mathbf{P}_n^m$, $\mathbf{B}_n^m$, and $\mathbf{C}_n^m$. The Hertz vector is defined at the field point $(\mathbf{r},t)$, and will be written as the expansion

$$\boldsymbol{\Pi} = \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} \Big\{ \Pi_{r,n}^m(r,t)\, \mathbf{P}_n^m(\theta,\phi) + \Pi_{B,n}^m(r,t)\, \mathbf{B}_n^m(\theta,\phi) + \Pi_{C,n}^m(r,t)\, \mathbf{C}_n^m(\theta,\phi) \Big\} , \qquad (3.1)$$

while the free-charge polarization vector is defined at the source point $(\mathbf{r}_1,t_1)$, and has the expansion

$$\mathbf{P} = \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} \Big\{ P_{r,n}^m(r_1,t_1)\, \mathbf{P}_n^m(\theta_1,\phi_1) + P_{B,n}^m(r_1,t_1)\, \mathbf{B}_n^m(\theta_1,\phi_1) + P_{C,n}^m(r_1,t_1)\, \mathbf{C}_n^m(\theta_1,\phi_1) \Big\} . \qquad (3.2)$$

The current density, $\mathbf{J}$, has an expansion of just the form (3.2), but involving the expansion coefficients $J_{r,n}^m(r_1,t_1)$, $J_{B,n}^m(r_1,t_1)$, and $J_{C,n}^m(r_1,t_1)$.

From the definitions of the vector spherical harmonics (in Ref. 2) it can be seen that $\underset{\sim}{B}_0^0$ and $\underset{\sim}{C}_0^0$ are both identically zero, so that for $n = 0$ only the terms involving $\underset{\sim}{P}_0^0$ are nonvanishing. The definitions also show that the vector spherical harmonics which differ only in the algebraic sign of <u>m</u> are complex conjugates. Since the vector functions $\underset{\sim}{J}$, $\underset{\approx}{P}$, and $\underset{\sim}{\Pi}$ are real quantities, not complex quantities (in this time-domain analysis), it consequently follows that the expansion coefficients which differ only in the sign of <u>m</u> are also complex conjugates, as illustrated by

$$\underset{\sim}{P}_{r,n}^{-m}(r_1,t_1) = \left[\underset{\sim}{P}_{r,n}^{+m}(r_1,t_1)\right]^* . \qquad (3.3)$$

4. DEFINITION OF θ' AND ϕ'

It will be convenient to define a polar coordinate system whose polar axis lies along the field-point vector $\underset{\sim}{r}$. In terms of this coordinate frame, the source-point vector $\underset{\sim}{r}_1$ has the polar coordinates (r_1, θ', ϕ'), and the source-point volume element is given by

$$dV_1 \;=\; r_1^{\,2} \sin\theta' \, dr_1 \, d\theta' \, d\phi' . \tag{4.1}$$

The polar angle θ' is the angle included between the vectors $\underset{\sim}{r}$ and $\underset{\sim}{r}_1$, so that

$$\underset{\sim}{r} \cdot \underset{\sim}{r}_1 \;=\; r\, r_1 \cos\theta' , \tag{4.2}$$

and the azimuth angle ϕ' will be defined as shown in Fig. 3 of Reference 6. The trigonometric transformation which expresses (θ_1, ϕ_1) in terms of (θ, ϕ) and (θ', ϕ') is given as Eqs. (3.5) of Ref. 6.

In this coordinate system, the Hertz vector (2.3) takes the form

$$\underset{\sim}{\Pi}(\underset{\sim}{r}, t) \;=\; \frac{1}{4\pi\epsilon_0} \int_{r_1=0}^{r_1=\infty} r_1^{\,2}\, dr_1 \int_{\theta'=0}^{\theta'=\pi} \frac{1}{s} \sin\theta' \, d\theta' \int_{\phi'=0}^{\phi'=2\pi} \underset{\sim}{P}(\underset{\sim}{r}_1, t-s/c)\, d\phi' . \tag{4.3}$$

5. INTEGRATION OVER ϕ'

In the expression for the Hertz vector, Eq. (4.3), the integration over the azimuth angle ϕ' will be carried out first. The geometrical configuration is shown in Fig. 1 of Reference 2. The source-point vector, $\underset{\sim}{r}_1$, is maintained at the constant length, r_1, and is swung about the field-point vector, $\underset{\sim}{r}$, with the angle θ' between these two vectors held constant.

The integration over ϕ' is a full circuit, from $\phi' = 0$ to $\phi' = 2\pi$, and in this integration the distance $\underline{s}$, defined in (2.4), remains constant (since θ' is constant). Thus in the integrand of the ϕ'-integration only the source-point angles (θ_1, ϕ_1) will vary. By (3.2) it can be seen that the variation is therefore confined to the vector spherical harmonics themselves, and that the expansion coefficients will remain constant.

The theorems of Reference 2 can now be brought into the analysis, and utilized in the ϕ'-integration of (4.3). In this way all nine components of the three vector spherical harmonics for a given (n,m) can be integrated over ϕ', giving explicit functions of $\cos\theta'$, multiplying vector spherical harmonics of (θ, ϕ).

6. INTEGRATION OVER θ'

After the integration over ϕ' has been done, the integration over θ' takes the general form

$$F(r,r_1,t) = \int_{\theta'=0}^{\theta'=\pi} \frac{1}{s} g(\cos\theta') f(r_1,t-s/c) \sin\theta' \, d\theta' , \qquad (6.1)$$

where <u>f</u> is one of the expansion coefficients in (3.2) and <u>g</u> is an explicit function obtained from one of the six integral theorems in Reference 2.

During the integration (6.1), the distances r and r_1 are held constant, but the distance s changes as θ' changes. By the law of cosines, the relationship (2.4) can be written as

$$s^2 = r^2 + r_1^2 - 2rr_1\cos\theta' , \qquad (6.2)$$

and its differentiation, with r and r_1 held constant, gives

$$s\,ds = rr_1 \sin\theta' \, d\theta' . \qquad (6.3)$$

Eq. (6.1) can thus be replaced by

$$F(r,r_1,t) = \int_{s=|r-r_1|}^{s=(r+r_1)} \frac{1}{rr_1} g(\xi) f(r_1,t-s/c) \, ds , \qquad (6.4)$$

where tho quantity ξ is defined by

$$\xi = \cos\theta' = \frac{1}{2\,r\,r_1}(r^2 + r_1^2 - s^2)\,. \tag{6.5}$$

The integration (6.4) can be rewritten also as an integration over the source-point time variable, t_1, through

$$t_1 = t - s/c\,, \tag{6.6}$$

$$dt_1 = -\frac{1}{c}\,ds\,. \tag{6.7}$$

In this case the quantity ξ is expressed as a function of t_1 having the form ξ_t, where

$$\xi_t = \frac{1}{2\,r\,r_1}\left[r^2 + r_1^2 - c^2(t-t_1)^2\right]. \tag{6.8}$$

The integration then has the appearance

$$F(r,r_1,t) = \int_{t_1=t-\frac{1}{c}(r+r_1)}^{t_1=t-\frac{1}{c}|r-r_1|} \frac{c}{r\,r_1}\,g(\xi_t)\,f(r_1,t_1)\,dt_1\,. \tag{6.9}$$

The function $g(\xi)$, obtained from the integral theorems of Reference 2, is in each case a polynomial in ξ, and through (6.5) it is therefore an even polynomial in s. For each such function, an associated function $G(r,r_1,s)$ can be defined by

$$G(r,r_1,s) = \int_{s=0}^{s=s} g(\xi)\, ds\,, \tag{6.10}$$

and this associated function is now an odd polynomial in $\underline{s}$.

The integration over Θ', as transformed by (6.4) into an integration over $\underline{s}$, will be carried out as an integration by parts, with the aid of (6.10). The result is

$$F(r,r_1,t) = \left[\frac{1}{r\,r_1} G(r,r_1,s)\, f(r_1,t-s/c)\right]_{s=|r-r_1|}^{s=(r+r_1)}$$
$$- \int_{s=|r-r_1|}^{s=(r+r_1)} \frac{1}{r\,r_1} G(r,r_1,s) \left[\frac{\partial}{\partial s} f(r_1,t-s/c)\right] ds\,. \tag{6.11}$$

In each case to be considered, the function $f(r_1,t_1)$ represents an expansion coefficient which has the time dependence of a component of the free-charge polarization vector, $\underset{\sim}{P}$. The time derivative of such a component gives the corresponding component of the source-current density, $\underset{\sim}{J}$, as shown by (2.1). Thus the derivative $\partial f/\partial s$ in (6.11) actually represents a current component. An example is given by

$$\frac{\partial}{\partial s} P^m_{r,n}(r_1,t-s/c) = -\frac{1}{c} J^m_{r,n}(r_1,t-s/c)\,. \tag{6.12}$$

Similarly, where f appears in (6.11) undifferentiated, representing for example the coefficient $\underline{P}^m_{r,n}$, it can be written as a time integral of the coefficient $J^m_{r,n}$:

$$\underline{P}^m_{r,n}(r_1, t-s/c) = \int_{t_1=t_0}^{t_1=t-s/c} J^m_{r,n}(r_1, t_1)\, dt_1 , \qquad (6.13)$$

where t_o is to be chosen as a time which precedes any current flow. (As mentioned in connection with Eq. (2.2), any pre-existing static field, formed by an earlier charge displacement which is not included in the transient current that is being analyzed, can be represented by a constant of integration, added to the right-hand side of (6.13).)

7. MATRIX REPRESENTATION

After the integration over $\wp'$, and the transformation from (6.1) to (6.4), but before the integration over $\underline{s}$, the Hertz-vector expansion coefficients can be represented compactly through the matrix equation:

$$\prod^{m}_{\sigma,n}(r,t) = \frac{1}{2\epsilon_0} \int_{r_1=0}^{r_1=\infty} (r_1/r)\,dr_1 \int_{s=|r-r_1|}^{s=(r+r_1)} g^{(n)}_{\sigma,\lambda}(\xi)\; \underset{\sim}{P}^{m}_{\lambda,n}(r_1,t-s/c)\,ds \; . \tag{7.1}$$

In (7.1) each of the Greek indices, σ and λ, runs through the three values: $\underline{r}$, $\underline{B}$, $\underline{C}$. A repeated Greek index indicates summation over these values. Thus (7.1) represents three equations, each of which may involve three expansion coefficients for the free-charge polarization vector $\underset{\sim}{P}$.

However, some of the elements of the matrix $g^{(n)}_{\sigma,\lambda}(\xi)$ are zero, so that the equations are in fact relatively simple. The matrix elements, determined through the use of the theorems in Reference 2, are:

$$g^{(n)}_{r,r}(\xi) = \xi\, P_n(\xi) \; , \tag{7.2a}$$

$$g^{(n)}_{B,B}(\xi) = \xi\, P_n(\xi) + \frac{(1-\xi^2)}{n(n+1)} \frac{d}{d\xi} P_n(\xi) \; , \tag{7.2b}$$

$$g^{(n)}_{C,C}(\xi) = P_n(\xi) \; , \tag{7.2c}$$

$$g^{(n)}_{r,B}(\xi) = g^{(n)}_{B,r}(\xi) = \frac{(1-\xi^2)}{\sqrt{n(n+1)}}\frac{d}{d\xi}P_n(\xi)\ , \qquad (7.2d)$$

$$g^{(n)}_{r,C}(\xi) = g^{(n)}_{C,r}(\xi) = 0\ , \qquad (7.2e)$$

$$g^{(n)}_{B,C}(\xi) = g^{(n)}_{C,B}(\xi) = 0\ . \qquad (7.2f)$$

When the integrals having the form (6.4) are integrated by parts, as illustrated in (6.11), then the Hertz-vector expansion coefficients (7.1) take the form:

$$\begin{aligned}
\Pi^m_{\sigma,n}(r,t) &= \frac{1}{2\epsilon_o}\int_{r_1=0}^{r_1=\infty}(r_1/r)\,dr_1\, G^{(n)}_{\sigma,\lambda}(r,r_1,r+r_1)\int_{t_1=t_o}^{t_1=t-(r+r_1)/c} J^m_{\lambda,n}(r_1,t_1)\,dt_1 \\
&- \frac{1}{2\epsilon_o}\int_{r_1=0}^{r_1=r}(r_1/r)\,dr_1\, G^{(n)}_{\sigma,\lambda}(r,r_1,r-r_1)\int_{t_1=t_o}^{t_1=t-(r-r_1)/c} J^m_{\lambda,n}(r_1,t_1)\,dt_1 \\
&- \frac{1}{2\epsilon_o}\int_{r_1=r}^{r_1=\infty}(r_1/r)\,dr_1\, G^{(n)}_{\sigma,\lambda}(r,r_1,r_1-r)\int_{t_1=t_o}^{t_1=t-(r_1-r)/c} J^m_{\lambda,n}(r_1,t_1)\,dt_1 \\
&+ \frac{1}{2\epsilon_o}\int_{r_1=0}^{r_1=r}(r_1/r)\,dr_1\int_{t_1=t-(r+r_1)/c}^{t_1=t-(r-r_1)/c} G^{(n)}_{\sigma,\lambda}(r,r_1,ct-ct_1)\, J^m_{\lambda,n}(r_1,t_1)\,dt_1 \\
&+ \frac{1}{2\epsilon_o}\int_{r_1=r}^{r_1=\infty}(r_1/r)\,dr_1\int_{t_1=t-(r+r_1)/c}^{t_1=t-(r_1-r)/c} G^{(n)}_{\sigma,\lambda}(r,r_1,ct-ct_1)\, J^m_{\lambda,n}(r_1,t_1)\,dt_1\ . \qquad (7.3
\end{aligned}$$

The matrix elements $G^{(n)}_{\sigma,\lambda}$ are related to the matrix elements $g^{(n)}_{\sigma,\lambda}$ through an equation of the form (6.10):

$$G^{(n)}_{\sigma,\lambda}(r,r_1,s) = \int_{s=0}^{s=s} g^{(n)}_{\sigma,\lambda}(\xi)\, ds \ , \qquad (7.4)$$

and can therefore be obtained explicitly with the aid of (7.2).

8. SUMMARY

Equation (7.3), together with (3.1) and (3.2), gives the Hertz vector $\underset{\sim}{\Pi}(\underset{\sim}{r},t)$ in terms of the source-current density function $\underset{\sim}{J}(\underset{\sim}{r}_1,t_1)$. The use of the expansion in terms of vector spherical harmonics, for both the Hertz vector and the source-current density, has provided a mode separation, in which a particular source mode, characterized by (n,m), leads to a field mode which is also characterized by (n,m). A source current with the symmetry of the vector spherical harmonic $\underset{\sim}{C}_n^m$ leads to a field mode with this same symmetry, but there is cross-coupling between source and field modes having the symmetries of the vector spherical harmonics $\underset{\sim}{P}_n^m$ and $\underset{\sim}{B}_n^m$.

The retarded Hertz vector, $\underset{\sim}{\Pi}(\underset{\sim}{r},t)$, is expressed as an integral over the source current, $\underset{\sim}{J}(\underset{\sim}{r}_1,t_1)$, within the region of space-time which is consistent with causality requirements. Since the original form of the Hertz vector, given in Eq. (4.3), is in accord with the requirements of causality, and since there is no contamination by 'advanced' fields, either in (4.3) or in any later stages of the analysis, it can be concluded that

the Hertz vector in the form (7.3), though expressed in the (r,t)-plane where causality requirements are not very transparent, will nevertheless remain fully consistent with the physical requirement that the electromagnetic effect of a moving charged particle should travel at the velocity of light if the medium is the vacuum.

The explicit calculation of the electric and magnetic fields, from the Hertz vector given here in (7.3) and (3.1), will be carried out in two following articles.[3,4]

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Part V. Magnetic Field Generated by a Transient Current Distribution . . . Roger E. Clapp and Luc Huang

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ABSTRACT

The preceding article gave the retarded Hertz vector in terms of a general transient source-current distribution. The Hertz vector is here differentiated to give the vector potential and the magnetic field vector. All of these vector quantities have been expanded in vector spherical harmonics, and in each case it is the expansion coefficients, which are functions of the radial distance r and the time t, which are expressed as two-dimensional integrals, in the (r,t)-plane, over the source-current expansion coefficients having the corresponding values of n and m, where these are the mode parameters characterizing the vector spherical harmonics, $\underline{P}_n^m(\theta,\phi)$, $\underline{B}_n^m(\theta,\phi)$, and $\underline{C}_n^m(\theta,\phi)$.

1. INTRODUCTION

In the previous article[1] the retarded Hertz vector was obtained, for a fully general transient source-current distribution. The source current, $\underset{\sim}{J}(\underset{\sim}{r}_1,t_1)$, was expressed as an expansion in vector spherical harmonics, with the dependence upon the angles (θ_1,ϕ_1) contained within these harmonics, while the dependence upon the radial distance r_1 and the source time t_1 was contained within the expansion coefficients. Similarly, the Hertz vector, $\underset{\sim}{\Pi}(\underset{\sim}{r},t)$, was expressed as an expansion in vector spherical harmonics which were functions of the angles (θ,ϕ), with expansion coefficients which were functions of the radial distance r and the time t.

It was found that, for a given (m,n), labeling the vector spherical harmonics, the expansion coefficients for the Hertz vector could be expressed in terms of the expansion coefficients for the source current. There was no coupling between the expansion coefficients for different choices of $\underline{n}$ or for different choices of $\underline{m}$. There was, however, coupling between two of the three vector harmonics, those denoted by $\underset{\sim}{P}_n^m$ and $\underset{\sim}{B}_n^m$, but no cross-coupling between either of these two and the third vector harmonic, denoted by $\underset{\sim}{C}_n^m$.

In the present article, this Hertz vector will be substituted into the equation

$$\underset{\sim}{H} = \epsilon_0 \, \underset{\sim}{\nabla} \times \frac{\partial}{\partial t} \underset{\sim}{\Pi} , \qquad (1.1)$$

which gives the magnetic field $\underset{\sim}{H}$ in terms of the Hertz vector $\underset{\sim}{\Pi}$, in MKS units for which

$$\epsilon_0 = 8.854 \cdot 10^{-12} \text{ farad/meter} . \qquad (1.2)$$

An alternative formulation will also be given, in terms of the vector potential, $\underset{\sim}{A}$, defined here by the equation

$$\underset{\sim}{A} = \frac{1}{c^2} \frac{\partial}{\partial t} \underset{\sim}{\Pi} . \qquad (1.3)$$

For the quasi-vacuum conditions that have been postulated, the magnetic field $\underset{\sim}{H}$ is then given by

$$\underset{\sim}{H} = \frac{1}{\mu_0} \underset{\sim}{\nabla} \times \underset{\sim}{A} , \qquad (1.4)$$

where

$$\frac{1}{\mu_0} = \epsilon_0 \, c^2 . \qquad (1.5)$$

2. MAGNETIC FIELD EXPANSION

If the Hertz vector and the magnetic field vector are both expressed in polar-coordinate components, then the vector equation (1.1) separates into the component equations:

$$H_r = \frac{\epsilon_0}{r\sin\theta}\left\{-\frac{\partial}{\partial\phi}\frac{\partial}{\partial t}\Pi_\theta + \frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial t}\Pi_\phi\right]\right\}, \qquad (2.1a)$$

$$H_\theta = \frac{\epsilon_0}{r}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\frac{\partial}{\partial t}\Pi_r - \frac{\partial}{\partial r}\left[r\frac{\partial}{\partial t}\Pi_\phi\right]\right\}, \qquad (2.1b)$$

$$H_\phi = \frac{\epsilon_0}{r}\left\{-\frac{\partial}{\partial\theta}\frac{\partial}{\partial t}\Pi_r + \frac{\partial}{\partial r}\left[r\frac{\partial}{\partial t}\Pi_\theta\right]\right\}. \qquad (2.1c)$$

However, it will be more convenient, and will lead to field expressions which are more compact, if the magnetic field vector is first expanded in terms of the vector spherical harmonics defined in Section 2 of Reference 2. This expansion can be written in the form:

$$\underset{\sim}{H}(\underset{\sim}{r},t) = \sum_{n=1}^{\infty}\sum_{m=-n}^{+n}\left\{H^m_{r,n}(r,t)\,\underset{\sim}{P}^m_n(\theta,\phi) + H^m_{B,n}(r,t)\,\underset{\sim}{B}^m_n(\theta,\phi) + H^m_{C,n}(r,t)\,\underset{\sim}{C}^m_n(\theta,\phi)\right\}. \qquad (2.2)$$

This summation begins with $n = 1$ because the spherically symmetric current component, with $n = 0$, does not give rise to any component of magnetic field, but only to a spherically symmetric, radially-directed electric field.

The expansion (2.2) parallels the similar expansions for the Hertz vector and for the current-density vector, as described in Section 3 of Reference 1. As in the earlier expansions, the reality of the field vector requires that expansion coefficients which differ only in the sign of m should be complex conjugates of each other, as in the example:

$$H^{-m}_{r,n}(r,t) = \left[H^{+m}_{r,n}(r,t)\right]^{*} . \tag{2.3}$$

3. TIME DERIVATIVE OF HERTZ VECTOR

When the Hertz vector, in the matrix form in Equation (7.1) of Reference 1, is differentiated with respect to the observation time t, the result is

$$\frac{\partial}{\partial t}\Pi^{m}_{\sigma,n}(r,t) = \frac{1}{2\epsilon_0}\int_{r_1=0}^{r_1=\infty}(r_1/r)\,dr_1\int_{s=|r-r_1|}^{s=(r+r_1)} g^{(n)}_{\sigma,\lambda}(\zeta)\; J^{m}_{\lambda,n}(r_1,t-s/c)\,ds\,. \tag{3.1}$$

The notation here is the same as the notation in Reference 1, with Greek subscripts running through the three values r, B, C, associated with the three vector spherical harmonics, $\underset{\sim}{P}^{m}_{n}$, $\underset{\sim}{B}^{m}_{n}$, $\underset{\sim}{C}^{m}_{n}$. As in (2.2), the dependence upon time and radial distance lies with the expansion coefficients, so that when the time derivative is taken it is only necessary to differentiate these expansion coefficients, as was done here in (3.1).

When the integration over ds in (3.1) is replaced by an integration over dt_1, with the use of (6.4-9) of Reference 1, then (3.1) takes the form

$$\frac{\partial}{\partial t}\Pi^{m}_{\sigma,n}(r,t) = \frac{c}{2\epsilon_0}\,(\iint)\,(r_1/r)\; g^{(n)}_{\sigma,\lambda}(\zeta_t)\; J^{m}_{\lambda,n}(r_1,t_1)\,dt_1\,dr_1\,, \tag{3.2}$$

where the abbreviation $(\iint)$ has the following equivalent meanings:

$$(\iint) = \int_{r_1=0}^{r_1=\infty} \int_{t_1=t-(r+r_1)/c}^{t_1=t-|r-r_1|/c} , \tag{3.3a}$$

$$(\iint) = \int_{t_1=-\infty}^{t_1=t} \int_{r_1=|r-c(t-t_1)|}^{r_1=r+c(t-t_1)} , \tag{3.3b}$$

$$(\iint) = \int_{r_1=0}^{r_1=r} \int_{t_1=t-(r+r_1)/c}^{t_1=t-(r-r_1)/c} + \int_{r_1=r}^{r_1=\infty} \int_{t_1=t-(r+r_1)/c}^{t_1=t-(r_1-r)/c} , \tag{3.3c}$$

$$(\iint) = \int_{t_1=-\infty}^{t_1=t-r/c} \int_{r_1=-r+c(t-t_1)}^{r_1=r+c(t-t_1)} + \int_{t_1=t-r/c}^{t_1=t} \int_{r_1=r-c(t-t_1)}^{r_1=r+c(t-t_1)} , \tag{3.3d}$$

which define the realm of integration in the (r_1, t_1)-plane.

4. VECTOR IDENTITIES

While Morse and Feshbach[3] have given a number of mathematical identities involving vector spherical harmonics, these are not in the form that is needed here. For the present application, there are three curl identities,

$$\mathbf{\nabla}\times\mathbf{P}_n^m(\theta,\phi) = \frac{\sqrt{n(n+1)}}{r}\,\mathbf{C}_n^m(\theta,\phi) \,, \tag{4.1}$$

$$\mathbf{\nabla}\times\mathbf{B}_n^m(\theta,\phi) = -\frac{1}{r}\,\mathbf{C}_n^m(\theta,\phi) \,, \tag{4.2}$$

$$\mathbf{\nabla}\times\mathbf{C}_n^m(\theta,\phi) = \frac{1}{r}\,\mathbf{B}_n^m(\theta,\phi) + \frac{\sqrt{n(n+1)}}{r}\,\mathbf{P}_n^m(\theta,\phi) \,, \tag{4.3}$$

which can readily be generalized to the situation in which each vector spherical harmonic is multiplied by a scalar function of the form $Q(r,t)$. The generalized identities are

$$\mathbf{\nabla}\times\left[Q\,\mathbf{P}_n^m\right] = \frac{\sqrt{n(n+1)}}{r}\,Q\;{}_{n}^{m} \,, \tag{4.4}$$

$$\mathbf{\nabla}\times\left[Q\,\mathbf{B}_n^m\right] = -\frac{1}{r}\frac{\partial}{\partial r}(r\,Q)\;\mathbf{C}_n^m \,, \tag{4.5}$$

$$\mathbf{\nabla}\times\left[Q\,\mathbf{C}_n^m\right] = \frac{1}{r}\frac{\partial}{\partial r}(r\,Q)\;\mathbf{B}_n^m + \frac{\sqrt{n(n+1)}}{r}\,Q\,\mathbf{P}_n^m \,. \tag{4.6}$$

5. MAGNETIC FIELD

In the interest of a compact notation, the three vector spherical harmonics will be represented collectively by $\underset{\sim}{V}^m_{\sigma,n}(\theta,\phi)$, where the Greek index σ takes on the three values r, B, C, as in (3.1). The explicit definitions are:

$$\underset{\sim}{V}^m_{r,n} = \underset{\sim}{P}^m_n , \tag{5.1a}$$

$$\underset{\sim}{V}^m_{B,n} = \underset{\sim}{B}^m_n , \tag{5.1b}$$

$$\underset{\sim}{V}^m_{C,n} = \underset{\sim}{C}^m_n . \tag{5.1c}$$

In terms of this new notation, and with the summation convention for repeated Greek indices, (2.2) can be written compactly as:

$$\underset{\sim}{H}(\underset{\sim}{r},t) = \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} H^m_{\sigma,n}(r,t)\ \underset{\sim}{V}^m_{\sigma,n}(\theta,\phi) , \tag{5.2}$$

with similar expressions for the Hertz vector and its time derivative.

In particular, the time derivative of the Hertz vector can be given in the form

$$\frac{\partial}{\partial t}\underset{\sim}{\Pi} = \frac{c}{2\epsilon_o}\sum_{n,m} \underset{\sim}{V}^m_{\sigma,n} \; (\iint) \; (r_1/r)\, g^{(n)}_{\sigma,\lambda}(\zeta_t) \; J^m_{\lambda,n}(r_1,t_1) \; dt_1\, dr_1 \, . \quad (5.3$$

Substitution into (1.1), with the use of (1.4-6), gives:

$$H^m_{r,n}(r,t) = \frac{c}{2r^2}\sqrt{n(n+1)} \; (\iint) \; P_n(\zeta_t) \; r_1 \; J^m_{C,n}(r_1,t_1) \; dt_1 \; dr_1 \; , \quad (5.4)$$

$$H^m_{B,n}(r,t) = \frac{c}{2r}\frac{\partial}{\partial r}\left[(\iint) \; P_n(\zeta_t) \; r_1 \; J^m_{C,n}(r_1,t_1) \; dt_1\, dr_1\right] \; ; \quad (5.5)$$

$$H^m_{C,n}(r,t) = \frac{c}{2r}\sqrt{n(n+1)} \; (\iint) \; P_n(\zeta_t) \; J^m_{r,n}(r_1,t_1) \; dt_1\, dr_1$$

$$- \frac{c}{2r} \; (\iint) \; P_n(\zeta_t) \; \frac{\partial}{\partial r_1}\left[r_1 \; J^m_{B,n}(r_1,t_1)\right] dt_1\, dr_1 \; . \quad (5.6)$$

In Equation (5.6), the second expression on the right-hand side was obtained through an integration by parts, in which the form (3.3d) was used for the integral operator.

6. DISCUSSION

Equations (5.4-6) give the magnetic field expansion coefficients, which are to be inserted into (5.2). It can be seen that there is an element of symmetry in the dependence of the three magnetic-field components upon the three current components, but that the symmetry is not as conveniently expressed in matrix form as was the case for the Hertz vector itself, as given in Eq. (7.1) of Reference 1.

While the forms (5.5) and (5.6) were chosen here because of their compactness, there are other forms for these magnetic field components which avoid the use of the operators $\partial/\partial r$ and $\partial/\partial r_1$. These other forms can be obtained from (5.5) and (5.6) through integration by parts and through the carrying-out of indicated differentiations.

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Part VI. Electric Field Generated by a Transient Current Distribution . . . .

. . . Roger E. Clapp, Luo Huang, and H. Tung Li

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ABSTRACT

The retarded Hertz vector, obtained earlier, is differentiated to give the scalar potential and the electric field vector, associated with a general transient source-current distribution. The scalar and vector quantities are expanded in terms of scalar and vector spherical harmonics. For each mode, characterized by specific values for n and m in the expansions, the scalar potential and electric field vector are expressed as two-dimensional integrals, in the (r,t)-plane, over the causally-accessible portion of the source-current distribution.

1. INTRODUCTION

In the preceding article[1], expressions were obtained for the magnetic field components associated with a general transient current distribution. Similar expressions will now be derived for the electric field components, and as before these will be derived from the previously derived expressions for the components of the retarded Hertz vector[2].

In terms of the retarded Hertz vector $\underset{\sim}{\Pi}$, the electric field vector is given by the equation

$$\underset{\sim}{E} = \underset{\sim}{\nabla}\underset{\sim}{\nabla}\cdot\underset{\sim}{\Pi} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\underset{\sim}{\Pi} . \tag{1.1}$$

In terms of the more familiar scalar and vector potentials, the electric vector is

$$\underset{\sim}{E} = -\underset{\sim}{\nabla}\varphi - \frac{\partial}{\partial t}\underset{\sim}{A} . \tag{1.2}$$

The identification of the vector potential,

$$\underset{\sim}{A} = \frac{1}{c^2}\frac{\partial}{\partial t}\underset{\sim}{\Pi} , \tag{1.3}$$

has already been made, in (1.3) of Reference 1. A corresponding identification of the scalar potential, in terms of the Hertz vector, is:

$$\varphi = -\underset{\sim}{\nabla}\cdot\underset{\sim}{\Pi} . \tag{1.4}$$

2. SCALAR IDENTITIES

In analogy with the vector identities given in Section 4 of Reference 1, there are certain scalar identities which arise from the action of the divergence operator upon the vector spherical harmonics, and upon products of a scalar function, $Q(r,t)$, and the vector spherical harmonics.

The three basic divergence identities are

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{P}_n^m(\theta,\phi) = \frac{2}{r}\, X_n^m(\theta,\phi) \,, \tag{2.1}$$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{B}_n^m(\theta,\phi) = -\frac{\sqrt{n(n+1)}}{r}\, X_n^m(\theta,\phi) \,, \tag{2.2}$$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{C}_n^m(\theta,\phi) = 0 \,. \tag{2.3}$$

The scalar spherical harmonics, X_n^m, and the vector spherical harmonics, $\underset{\sim}{P}_n^m$, $\underset{\sim}{B}_n^m$, and $\underset{\sim}{C}_n^m$, have been defined in Section 2 of Reference 3.

The more general divergence identities, in which each vector spherical harmonic is multiplied by the scalar function $Q(r,t)$, are:

$$\underset{\sim}{\nabla}\cdot\left[Q\,\underset{\sim}{P}_n^m\right] = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\,Q)\;X_n^m\;, \tag{2.4}$$

$$\underset{\sim}{\nabla}\cdot\left[Q\,\underset{\sim}{B}_n^m\right] = -\,\frac{\sqrt{n(n+1)}}{r}\,Q\,X_n^m\;, \tag{2.5}$$

$$\underset{\sim}{\nabla}\cdot\left[Q\,\underset{\sim}{C}_n^m\right] = 0\;. \tag{2.6}$$

As will be seen, it is the vanishing of the divergence in (2.6) that can be associated with the fact that the source-current components involving the vector spherical harmonics $\underset{\sim}{C}_n^m$ do not lead to any charge accumulation, hence these currents do not give any electrostatic contribution to $\underset{\sim}{E}(\underset{\sim}{r},t)$. These currents are circulatory in character, and make only inductive and radiative field contributions.

3. DIVERGENCE OF HERTZ VECTOR

In the evaluation of the divergence of the Hertz vector, one of the functions defined in Section 7 of Reference 2 appears repeatedly. This is the function

$$G_{C,C}^{(n)}(r;r_1,s) = \int_{s=0}^{s=S} P_n(\xi)\, ds \,, \qquad (3.1)$$

where

$$\xi = \frac{1}{2\,r\,r_1}\,(r^2 + r_1^2 - s^2) \,. \qquad (3.2)$$

Special values of this function are

$$G_{C,C}^{(n)}(r,r_1,r+r_1) = (2n+1)^{-1}\,(r^{n+1}\,r_1^{-n} + r^{-n}\,r_1^{n+1}), \qquad (3.3a)$$

$$G_{C,C}^{(n)}(r,r_1,r-r_1) = (2n+1)^{-1}\,(r^{n+1}\,r_1^{-n} - r^{-n}\,r_1^{n+1}), \qquad (3.3b)$$

$$G_{C,C}^{(n)}(r,r_1,r_1-r) = (2n+1)^{-1}\,(-\,r^{n+1}\,r_1^{-n} + r^{-n}\,r_1^{n+1}). \qquad (3.3c)$$

Equations (3.3) are established in the Appendix.

The divergence of the Hertz vector does not involve the source currents with the $J^{m}_{C,n}$ symmetry, as noted earlier in connection with Eq. (2.6). However, both the $J^{m}_{r,n}$ and the $J^{m}_{B,n}$ current components make contributions to this divergence. The explicit form for this divergence is:

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{\Pi} = \Bigg\{ -\int_{r_1=0}^{r_1=\infty} r_1^2 \frac{\partial}{\partial r_1}\left[\frac{1}{r_1}\, G^{(n)}_{C,C}(r,r_1,r+r_1)\right] \int_{t_1=t_0}^{t_1=t-(r+r_1)/c} J^{m}_{r,n}(r_1,t_1)\, dt_1\, dr_1$$

$$+\int_{r_1=0}^{r_1=r} r_1^2 \frac{\partial}{\partial r_1}\left[\frac{1}{r_1}\, G^{(n)}_{C,C}(r,r_1,r-r_1)\right] \int_{t_1=t_0}^{t_1=t-(r-r_1)/c} J^{m}_{r,n}(r_1,t_1)\, dt_1\, dr_1$$

$$+\int_{r_1=r}^{r_1=\infty} r_1^2 \frac{\partial}{\partial r_1}\left[\frac{1}{r_1}\, G^{(n)}_{C,C}(r,r_1,r_1-r)\right] \int_{t_1=t_0}^{t_1=t-(r_1-r)/c} J^{m}_{r,n}(r_1,t_1)\, dt_1\, dr_1$$

$$-\,(\int\!\!\int)\, r_1^2 \frac{\partial}{\partial r_1}\left[\frac{1}{r_1}\, G^{(n)}_{C,C}(r,r_1,ct-ct_1)\right]\ {}^{m}{}_{n}(r_1,t_1)\, dt_1\, dr_1$$

$$-\sqrt{n(n+1)}\int_{r_1=0}^{r_1=\infty} G^{(n)}_{C,C}(r,r_1,r+r_1) \int_{t_1=t_0}^{t_1=t-(r+r_1)/c} J^{(n)}_{B,n}(r_1,t_1)\, dt_1\, dr_1$$

$$+ \sqrt{n(n+1)} \int_{r_1=0}^{r_1=r} G_{C,C}^{(n)}(r,r_1,r-r_1) \int_{t_1=t_o}^{t_1=t-(r-r_1)/c} J_{B,n}^{m}(r_1,t_1)\ dt_1\ dr_1$$

$$+ \sqrt{n(n+1)} \int_{r_1=r}^{r_1=\infty} G_{C,C}^{(n)}(r,r_1,r_1-r) \int_{t_1=t_o}^{t_1=t-(r_1-r)/c} J_{B,n}^{m}(r_1,t_1)\ dt_1\ dr_1$$

$$\left. - \sqrt{n(n+1)}\ (\iint)\ G_{C,C}^{(n)}(r,r_1,ct-ct_1)\ J_{B,n}^{m}(r_1,t_1)\ dt_1\ dr_1 \right\} \frac{1}{2\epsilon_o r}\ X_n^m(\theta,\phi). \tag{3.4}$$

The integral symbol $(\iint)$ has been defined in Eq. (3.3) of Reference 1.

4. ELECTRIC FIELD

The electric field vector is obtained from Eq. (1.1). This includes an operation in which the gradient of the divergence of the Hertz vector is taken, and for this operation it is convenient to make use of the identity

$$\underset{\sim}{\nabla}\left[Q(r,t)\ X_n^m(\theta,\phi)\right] = \left[\frac{\partial}{\partial r} Q(r,t)\right] \underset{\sim}{P}{}_n^m(\theta,\phi) + \sqrt{n(n+1)}\ Q(r,t)\ \underset{\sim}{B}{}_n^m(\theta,\phi) \ . \qquad (4.1)$$

The second time derivative of the Hertz vector also appears, and this is conveniently found from the time derivative of the expression given in Eq. (3.2) of Reference 1. After an integration by parts, this second time derivative is found to be expressible in the compact matrix form:

$$\frac{\partial^2}{\partial t^2} \Pi^m_{\sigma,n}(r,t) = \frac{c}{2\epsilon_0}\ (\iint)\ (r_1/r)\ g^{(n)}_{\sigma,\lambda}(\zeta_t)\left[\frac{\partial}{\partial t_1} J^m_{\lambda,n}(r_1,t_1)\right] dt_1\, dr_1 \qquad (4.2)$$

As before, a repeated Greek index indicates summation over the values r, B, C, and the functions $g^{(n)}_{\sigma,\lambda}$ are as defined in (7.2) of Reference 2.

When the operations indicated in Eq. (1.1) are carried out, the resulting expressions, for the expansion coefficients for the electric field vector, are:

$$E^{m}_{r,n}(r,t) = -\frac{1}{\epsilon_o}\int_{t_1=t_o}^{t_1=t} J^{m}_{r,n}(r,t_1)\, dt_1$$

$$+ \frac{n(n+1)}{(2n+1)\epsilon_o}\int_{r_1=0}^{r_1=r}\int_{t_1=t_o}^{t_1=t-(r-r_1)/c} r^{-n-2}\, r_1^{n+1}\; J^{m}_{r,n}(r_1,t_1)\, dt_1\, dr_1$$

$$+ \frac{n(n+1)}{(2n+1)\epsilon_o}\int_{r_1=r}^{r_1=\infty}\int_{t_1=t_o}^{t_1=t-(r_1-r)/c} r^{n-1}\, r_1^{-n}\; J^{m}_{r,n}(r_1,t_1)\, dt_1\, dr_1$$

$$- \frac{n(n+1)}{2(2n+1)\epsilon_o}(\iint)\,(r^{-n-2}\, r_1^{n+1} + r^{n-1}\, r_1^{-n})\; J^{m}_{r,n}(r_1,t_1)\, dt_1\, dr_1$$

$$+ \frac{n(n+1)}{2\,\epsilon_o r^2}(\iint)\; G^{(n)}_{C,C}(r,r_1,ct-ct_1)\; J^{m}_{r,n}(r_1,t_1)\, dt_1\, dr_1$$

$$+ \frac{(n+1)\sqrt{n(n+1)}}{(2n+1)\epsilon_o}\int_{r_1=0}^{r_1=r}\int_{t_1=t_o}^{t_1=t-(r-r_1)/c} r^{-n-2}\, r_1^{n+1}\; J^{m}_{B,n}(r_1,t_1)\, dt_1\, dr_1$$

$$- \frac{n\sqrt{n(n+1)}}{(2n+1)\,\epsilon_o} \int_{r_1=r}^{r_1=\infty} \int_{t_1=t_o}^{t_1=t-(r_1-r)/c} r^{n-1}\, r_1^{-n}\, J^m_{B,n}(r_1,t_1)\, dt_1\, dr_1$$

$$- \frac{\sqrt{n(n+1)}}{2(2n+1)\,\epsilon_o} (\iint) \left[(n+1)\, r^{-n-2}\, r_1^{n+1} - n\, r^{n-1}\, r_1^{-n}\right] J^m_{B,n}(r_1,t_1)\, dt_1\, dr_1$$

$$+ \frac{\sqrt{n(n+1)}}{2\,\epsilon_o\, r^2} (\iint) \left[r_1 \frac{\partial}{\partial r_1} G^{(n)}_{C,C}(r,r_1,ct-ct_1)\right] J^m_{B,n}(r_1,t_1)\, dt_1\, dr_1 \,, \quad (4.3)$$

$$E^m_{B,n}(r,t) = - \frac{n\sqrt{n(n+1)}}{(2n+1)\,\epsilon_o} \int_{r_1=0}^{r_1=r} \int_{t_1=t_o}^{t_1=t-(r-r_1)/c} r^{-n-2}\, r_1^{n+1}\, J^m_{r,n}(r_1,t_1)\, dt_1\, dr_1$$

$$+ \frac{(n+1)\sqrt{n(n+1)}}{(2n+1)\,\epsilon_o} \int_{r_1=r}^{r_1=\infty} \int_{t_1=t_o}^{t_1=t-(r_1-r)/c} r^{n-1}\, r_1^{-n}\, J^m_{r,n}(r_1,t_1)\, dt_1\, dr_1$$

$$+ \frac{\sqrt{n(n+1)}}{2(2n+1)\,\epsilon_o} (\iint) \left[n\, r^{-n-2}\, r_1^{n+1} - (n+1)\, r^{n-1}\, r_1^{-n}\right] J^m_{r,n}(r_1,t_1)\, dt_1\, dr_1$$

$$+ \frac{\sqrt{n(n+1)}}{2\,\epsilon_o\, r^2} (\iint) \left[r \frac{\partial}{\partial r} G^{(n)}_{C,C}(r,r_1,ct-ct_1)\right] J^m_{r,n}(r_1,t_1)\, dt_1\, dr_1$$

$$- \frac{1}{2\epsilon_o c r} \int_{r_1=0}^{r_1=\infty} (-1)^n \, r_1 \, J^m_{B,n}(r_1, t-[r+r_1]/c) \, dr_1$$

$$- \frac{1}{2\epsilon_o c r} \int_{r_1=0}^{r_1=r} r_1 \, J^m_{B,n}(r_1, t-[r-r_1]/c) \, dr_1$$

$$- \frac{1}{2\epsilon_o c r} \int_{r_1=r}^{r_1=\infty} r_1 \, J^m_{B,n}(r_1, t-[r_1-r]/c) \, dr_1$$

$$- \frac{n(n+1)}{(2n+1)\epsilon_o} \int_{r_1=0}^{r_1=r} \int_{t_1=t_o}^{t_1=t-(r-r_1)/c} r^{-n-2} \, r_1^{n+1} \, J^m_{B,n}(r_1, t_1) \, dt_1 \, dr_1$$

$$- \frac{n(n+1)}{(2n+1)\epsilon_o} \int_{r_1=r}^{r_1=\infty} \int_{t_1=t_o}^{t_1=t-(r_1-r)/c} r^{n-1} \, r_1^{-n} \, J^m_{B,n}(r_1, t_1) \, dt_1 \, dr_1$$

$$+ \frac{n(n+1)}{2(2n+1)\epsilon_o} (\iint) \, (r^{-n-2} \, r_1^{n+1} + r^{n-1} \, r_1^{-n}) \, J^m_{B,n}(r_1, t_1) \, dt_1 \, dr_1$$

$$+ \frac{1}{2\epsilon_o r^2} (\iint) \left[r \, r_1 \frac{\partial^2}{\partial r \, \partial r_1} G^{(n)}_{C,C}(r, r_1, ct-ct_1) \right] J^m_{B,n}(r_1, t_1) \, dt_1 \, dr_1 \, ,$$

(4.4)

$$E^{m}_{C,n}(r,t) \;=\; -\,\frac{1}{2\,\epsilon_0\,c}\,(\iint)\;(r_1/r)\;P_n(\xi_t)\left[\frac{\partial}{\partial t_1}\,J^{m}_{C,n}(r_1,t_1)\right]dt_1\,dr_1\;. \tag{4.5}$$

5. DISCUSSION

Equations (4.3-5) give the components of the electric field, in terms of the components of the source current. These equations are not given in the general matrix form, since their relative complexity precludes for the present a simple matrix representation. However, there are elements of symmetry involved, and a compact matrix representation may eventually be devised.

The electric field components are here given as explicit integrations over the source-current components. For the inductive and radiative contributions, the integration is limited to the realm specified by the notation ($\iint$). This realm limits the source currents which are 'visible' at the observation point. However, in addition to these immediately-sensed contributions, there are electric field contributions which are electrostatic and are associated with earlier flow of current, establishing a distribution of electric charge which generates an electrostatic field. The contributions to the electric field which are associated with this electrostatic dipole-moment distribution are expressed in the integrals which have as their lower limit $t_1 = t_o$, where t_o is a time preceding any of the transient

current flow which contributes to the fields that are being calculated.

The important aspect of Equations (4.3-5) is their expression of the causal relationship between the source current, $\underset{\sim}{J}$, and the electric field, $\underset{\sim}{E}$, which is generated by this source current. When these equations are used as the basis for a numerical solution of Maxwell's equations, the requirements of causality will be met.[4]

APPENDIX

For the derivation of Equations (3.3), it will first be noted that the Legendre polynomial $P_n(\zeta)$, which can be written in the form

$$P_n(\zeta) = \frac{1}{2^n n!} \frac{d^n}{d\zeta^n}(\zeta^2 - 1)^n , \tag{A.1}$$

is an odd polynomial in ζ if n is odd, an even polynomial if n is even, so that

$$P_n(-\zeta) = (-1)^n P_n(\zeta) . \tag{A.2}$$

When the argument ζ is written out as

$$\zeta = \frac{1}{2 r r_1}(r^2 + r_1^2 - s^2) , \tag{A.3}$$

then the Legendre polynomial $P_n(\zeta)$ has the form

$$P_n(\zeta) = \frac{1}{(2 r r_1)^n} f_n(r^2, r_1^2, s^2) , \tag{A.4}$$

where the function f_n is a finite polynomial in its three arguments.

It follows that the integral

$$F_n(r,r_1,R) = \int_{s=0}^{s=R} P_n(\xi)\, ds \qquad (A.5)$$

is an odd polynomial in R, so that

$$F_n(r,r_1,-R) = - F_n(r,r_1,R) . \qquad (A.6)$$

It can also be shown, from the form of (A.4), that

$$F_n(-r,r_1,R) = (-1)^n F_n(r,r_1,R) , \qquad (A.7a)$$

$$F_n(r,-r_1,R) = (-1)^n F_n(r,r_1,R) , \qquad (A.7b)$$

$$F_n(-r,-r_1,R) = F_n(r,r_1,R) . \qquad (A.7c)$$

The three integrals to be evaluated are

$$F_n(r,r_1,r+r_1) = \int_{s=0}^{s=r+r_1} P_n(\xi)\, ds , \qquad (A.8a)$$

$$F_n(r,r_1,r-r_1) = \int_{s=0}^{s=r-r_1} P_n(\xi)\, ds , \qquad (A.8b)$$

$$F_n(r,r_1,r_1-r) = \int_{s=0}^{s=r_1-r} P_n(\xi)\, ds . \qquad (A.8c)$$

All three are defined here with no restrictions on the relative magnitudes of r and r_1. From (A.6) it can be seen that

$$F_n(r,r_1,r-r_1) = - F_n(r,r_1,r_1-r) , \tag{A.9a}$$

and from the symmetrical way that r and r_1 enter into ξ, in (A.3), it is also clear that

$$F_n(r_1,r,r_1-r) = - F_n(r,r_1,r-r_1) , \tag{A.9b}$$

$$F_n(r_1,r,r_1+r) = F_n(r,r_1,r+r_1) . \tag{A.9c}$$

Equation (A.3) can be solved for s, giving

$$s = (r^2 - 2 r r_1 \xi + r_1^2)^{1/2} . \tag{A.10}$$

When r and r_1 are both held constant, the differentials ds and $d\xi$ will be related by

$$s\,ds = - r r_1 d\xi . \tag{A.11}$$

If r_1 is smaller than r, the expansion

$$\frac{r}{s} = P_o(\xi) + \frac{r_1}{r} P_1(\xi) + \frac{r_1^2}{r^2} P_2(\xi) + \ldots \tag{A.12}$$

converges, and it is possible to write

$$F_n(r,r_1,r+r_1) - F_n(r,r_1,r-r_1) = \int_{s=r-r_1}^{s=r+r_1} P_n(\zeta)\, ds$$

$$= -\int_{\zeta=+1}^{\zeta=-1} s^{-1}\, r\, r_1\, P_n(\zeta)\, d\zeta = r_1 \int_{\zeta=-1}^{\zeta=+1} P_n(\zeta)\, \frac{r}{s}\, d\zeta = r_1\, \frac{r_1^{\,n}}{r^n}\, \frac{2}{(2n+1)} . \quad (A.13)$$

In (A.13) the series (A.12) has been substituted, and the orthogonality properties of the Legendre polynomials have been utilized.

If, on the other hand, r_1 is greater than r, then the convergent series expansion is

$$\frac{r_1}{s} = P_o(\zeta) + \frac{r}{r_1} P_1(\zeta) + \frac{r^2}{r_1^{\,2}} P_2(\zeta) + \dots , \qquad (A.14)$$

and the resulting equation, analogous to (A.13), is

$$F_n(r,r_1,r+r_1) - F_n(r,r_1,r_1-r) = r\, \frac{r^n}{r_1^{\,n}}\, \frac{2}{(2n+1)} . \qquad (A.15)$$

From (A.4) and (A.5) it is apparent that the multiplication of $F_n(r,r_1,R)$ by $(2\, r\, r_1)^n$ will cancel the factors in the denominator and give a finite polynomial. In particular, if Equation (A.13) is multiplied on both sides by $(2\, r\, r_1)^n$, it will become a relationship between the sum of two finite polynomials (on the left) and a

monomial (on the right). Evidently all of the terms in the two polynomials must subtract out, except for the terms which add to give the monomial on the right. Moreover, this algebraic relationship involving finite polynomials, while established for r_1 less than r, can be continued analytically to include the regions where r_1 is greater than r. Similarly, Equation (A.15), while established for r_1 greater than r, is a simple algebraic relationship which can be continued analytically to the realm where r_1 is less than r.

When (A.9a) is inserted into (A.15), the result is

$$F_n(r,r_1,r+r_1) + F_n(r,r_1,r-r_1) = r\,\frac{r^n}{r_1^{\,n}}\,\frac{2}{(2n+1)}\,. \qquad \text{(A.16)}$$

Now, from (A.13), (A.16), and (A.9a), the desired results are obtained:

$$F_n(r,r_1,r+r_1) = (2n+1)^{-1}\,(r^{n+1}\,r_1^{\,-n} + r^{-n}\,r_1^{\,n+1})\,, \qquad \text{(A.17a)}$$

$$F_n(r,r_1,r-r_1) = (2n+1)^{-1}\,(r^{n+1}\,r_1^{\,-n} - r^{-n}\,r_1^{\,n+1})\,, \qquad \text{(A.17b)}$$

$$F_n(r,r_1,r_1-r) = (2n+1)^{-1}\,(-\,r^{n+1}\,r_1^{\,-n} + r^{-n}\,r_1^{\,n+1})\,. \qquad \text{(A.17c)}$$

These results are equivalent to Equations (3.3), since it is apparent from the definition of F_n in (A.5) that

$$F_n(r,r_1,R) = G^{(n)}_{C,C}(r,r_1,R)\,. \qquad \text{(A.18)}$$

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Part VII. Causal and Non-Causal Numerical Solutions of Maxwell's Equations . . . Roger E. Clapp

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ABSTRACT

Maxwell's equations in differential form do not distinguish between advanced and retarded solutions. Unless special precautions are taken, a point-by-point numerical integration, using a finite-difference version of Maxwell's equations, will lead to a mixture of advanced and retarded fields, which is inadmissible on physical grounds. The causality requirement can be met if Maxwell's equations are written in integral-equation form, with retardation incorporated in all the integrands. A solution using numerical integration then will be physically acceptable. The distinction between causal and non-causal solutions is illustrated by an example in which the problem symmetry permits separation of variables and the reduction of the four-dimensional space-time problem to a two-dimensional problem involving only radial distance $\underline{r}$ and time $\underline{t}$.

1. INTRODUCTION

Maxwell's equations, as they stand, admit of two kinds of exact solutions for the electromagnetic fields generated by a specified time-dependent distribution of electric charge and current. These are the retarded solution and the advanced solution. An example of a retarded solution is the Liénard-Wiechert solution for the fields generated by a moving charged particle.[1] The fields at an observation point are expressed in terms of the position, velocity, and acceleration of the particle at a time which is earlier than the observation time, by an amount sufficient to allow for propagation at the velocity c from the particle to the point of observation.

Similar ~~The same~~ equations can be used for the corresponding advanced solution, if the particle's position, velocity, and acceleration are taken from a later point along its trajectory, where the particle time, t_1, and the observation time, t, are related by

$$t_1 = t + s/c \; , \qquad (1.1)$$

with s the propagation distance. In this case, the radiated fields are received before they have been emitted.

It is obvious, of course, that the advanced solution, involving (1.1), is to be excluded on physical grounds, and that only the retarded solution is to be considered as satisfying causality requirements. However, both are exact solutions of Maxwell's equations, so that the selection of one over the other must be made on the basis of an auxiliary condition or requirement, which is not explicitly or implicitly contained in Maxwell's equations themselves. Without this auxiliary condition, a general solution for the electromagnetic field associated with the moving charged particle would be a superposition of the advanced and retarded solutions, with an adjustable parameter specifying the proportions in which these two solutions entered the general solution.

In cases where exact solutions of Maxwell's equations are not available, numerical methods can be used, based on a finite-difference analog of the differential equations. However, the problem of causality will arise here again. Since the differential equations do not distinguish between advanced and retarded solutions, the finite-difference analogs of these equations will be unable to make any such distinction. As in the case of the differential equations, an auxiliary condition must be imposed, to select the retarded solution.

There is a class of difficult but important problems, in which an imposed primary current of transient form moves through and near a region of transient conductivity. The fields from the primary current induce a secondary current in the region of finite conductivity, and this secondary current makes its own contribution to the local fields. Hence the secondary current will itself be modified as a consequence of its own radiation. An integral equation for the secondary current has then to be solved. However, all of the field contributions, made by primary current and by secondary current, must be retarded fields, if the causality requirement is to be satisfied.

While a problem of this form is in the general case expressed as a four-dimensional, space-time integral equation, there are certain special symmetry conditions, of importance in practical examples, which permit the problem to be separated in polar coordinates, and reduced to a two-dimensional problem involving only a radial distance and the time variable. Unfortunately, the reduction to two dimensions further obscures the causality condition, if the set of differential equations is transformed directly and then replaced by the corresponding set of finite-difference equations. An attempt to solve these equations numerically[2] leads to a serious violation of causality requirements, as will be shown later.

A finite-step procedure which utilizes a set of lattice points in the (r,t)-plane, but which satisfies causality requirements at every step, is also described here. This is a lattice version, or finite-difference version, of an integral method developed for the solution of electromagnetic transient problems.[3] This method, like the differential metho makes use of an expansion in vector spherical harmonics, for the source currents and the field vectors.[3]

The lattice version of the integral method is an iterative solution of the integral equation for the secondary current. The iteration progresses outward in radial distance, and forward in time, within the problem domain in the (r,t)-plane. At each step the local electric field is computed, as a summation of the contributions of primary and secondary currents previously computed, with retardation fully accounted for. The product of the local electric field and the local conductivity function (a known quantity) is then equal to the secondary current density at that lattice point. This secondary current is then added to the primary current at that lattice point (a known quantity). The iteration now progresses to the next point in the lattice, and so on, until the full distribution of the secondary current, over the (r,t)-plane, has been determined.

2. SECONDARY CURRENT

The causality problem is illustrated most clearly if the transient conductivity function does not depend on angle, but only on radial distance and time:

$$\sigma = \sigma(r,t) \quad (2.1)$$

It will further be assumed that the secondary current is strictly the product of the local electric field and the local conductivity, independent of the magnitude of the local magnetic field. In other words, the Hall coefficient for the problem region will be assumed to be zero.

Under these simplifying conditions, the current density can be written as

$$\underset{\sim}{J} = \underset{\sim}{K} + \sigma(r,t)\,\underset{\sim}{E} , \quad (2.2)$$

where $\underset{\sim}{K}$ is the primary current. For the particular problem toward which this analytical work has been directed, the calculation of the electromagnetic radiation associated with a nuclear detonation in the lower atmosphere[4], the primary current $\underset{\sim}{K}$ consists of Compton electrons ejected from air molecules by gamma-rays from the nuclear reaction.

The conductivity $\sigma(r,t)$ is due to the momentary presence in the air of free electrons, released in the ionizing collisions which slow these Compton electrons, plus the later presence of heavier ions, such as the O_2^- ions formed when these free electrons become attached to oxygen molecules in the air, plus the positive ions left behind when the Compton electrons were ejected.

While some bending of the trajectories of these charged particles is to be expected, in the transient magnetic field that is generated by their motion, it will be assumed (as noted above) that the secondary current component generated by this bending is negligible in comparison with the current $\underset{\sim}{J}$ given in (2.2). ¶ The advantage of these simplifying assumptions is that there results a clean separation of the problem into a sequence of modes which are not coupled together. The integral equation for the secondary current separates into a set of uncoupled integral equations, one for each mode.

3. SEPARATION INTO MODES

When the conductivity function has the form (2.1), implying that the physical process to be analyzed has a center of at least partial symmetry, it is appropriate to make an expansion in spherical coordinates about this central point. In the example under consideration[4], the central point is the location of the nuclear detonation, which produces the primary current ~~through this~~ and/the transient conductivity.

The electric and magnetic fields, $\underset{\sim}{E}(\underset{\sim}{r},t)$ and $\underset{\sim}{H}(\underset{\sim}{r},t)$, will be expanded in terms of the vector spherical harmonics,[5] $\underset{\sim}{P}_n^m(\theta,\phi)$, $\underset{\sim}{B}_n^m(\theta,\phi)$, and $\underset{\sim}{C}_n^m(\theta,\phi)$. The angular dependence is contained within these harmonics, while the expansion coefficients contain the dependence upon radial distance, $\underline{r}$, and time, $\underline{t}$. The electric field, in particular, will have the expansion

$$\underset{\sim}{E}(\underset{\sim}{r},t) = \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} \Big\{ E_{r,n}^{m}(r,t)\, \underset{\sim}{P}_n^m(\theta,\phi) + E_{B,n}^{m}(r,t)\, \underset{\sim}{B}_n^m(\theta,\phi) + E_{C,n}^{m}(r,t)\, \underset{\sim}{C}_n^m(\theta,\phi) \Big\} . \qquad (3.1)$$

The magnetic field has an expansion of the same form, but with the expansion coefficients in (3.1) replaced by $H^m_{r,n}(r,t)$, $H^m_{B,n}(r,t)$, and $H^m_{C,n}(r,t)$. Similarly, the source current, $\underset{\sim}{J}$, and the primary current, $\underset{\sim}{K}$, will have expansions in terms of the same vector spherical harmonics, but it will sometimes be convenient to write the source current as $\underset{\sim}{J}(\underset{\sim}{r}_1,t_1)$, with an expansion involving such terms as $J^m_{r,n}(r_1,t_1)\ \underset{\sim}{P}^m_n(\theta_1,\phi_1)$, in order that the source point $(\underset{\sim}{r}_1,t_1)$ should be clearly distinguished from the field point $(\underset{\sim}{r},t)$.

When there is rotational symmetry about the vertical axis, so that the source current does not depend on the azimuth angle ϕ_1, then the fields will also be independent of ϕ, and the expansions such as (3.1) will be limited to those terms for which $m = 0$. Furthermore, if the source current consists of radial and polar components only, with no current circulating about the vertical axis, then the expansions will be limited still further. With these simplifications, the electric field (3.1) is reduced to:

$$\underset{\sim}{E}(\underset{\sim}{r},t) = \sum_{n=0}^{\infty} \left\{ E^0_{r,n}(r,t)\ \underset{\sim}{P}^0_n(\theta,\phi) + E^0_{B,n}(r,t)\ \underset{\sim}{B}^0_n(\theta,\phi) \right\}, \quad (3.2)$$

while the magnetic field takes the still simpler form:

$$\underset{\sim}{H}(\underset{\sim}{r},t) = \sum_{n=1}^{\infty} H^{0}_{C,n}(r,t)\, \underset{\sim}{C}^{0}_{n}(\theta,\phi) , \tag{3.3}$$

the term with $n = 0$ being omitted because the vector spherical harmonic $\underset{\sim}{C}^{0}_{0}$ is identically zero, as is $\underset{\sim}{B}^{0}_{0}$, though $\underset{\sim}{P}^{0}_{0}$ does not vanish.

The explicit forms for the vector spherical harmonics in (3.2) and (3.3) are

$$\underset{\sim}{P}^{0}_{n}(\theta,\phi) = \underset{\sim}{a}_{r}\, P_{n}(\cos\theta) , \tag{3.4}$$

$$\underset{\sim}{B}^{0}_{n}(\theta,\phi) = \frac{-\underset{\sim}{a}_{\theta}}{\sqrt{n(n+1)}}\, P^{1}_{n}(\cos\theta) , \tag{3.5}$$

$$\underset{\sim}{C}^{0}_{n}(\theta,\phi) = \frac{\underset{\sim}{a}_{\phi}}{\sqrt{n(n+1)}}\, P^{1}_{n}(\cos\theta) , \tag{3.6}$$

where $P_n(\cos\theta)$ are the Legendre polynomials, and

$$P^{1}_{n}(\cos\theta) = -\frac{d}{d\theta}\, P_{n}(\cos\theta)$$

$$= \sin\theta \frac{d}{d(\cos\theta)}\, P_{n}(\cos\theta) . \tag{3.7}$$

The unit vectors $\underset{\sim}{a}_{r}$, $\underset{\sim}{a}_{\theta}$, $\underset{\sim}{a}_{\phi}$ can be expressed in terms of the rectangular-coordinate unit vectors, $\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{k}$, through:

$$\mathbf{a}_r = \mathbf{i}\sin\theta\cos\phi + \mathbf{j}\sin\theta\sin\phi + \mathbf{k}\cos\theta\,,$$

$$\mathbf{a}_\theta = \mathbf{i}\cos\theta\cos\phi + \mathbf{j}\cos\theta\sin\phi - \mathbf{k}\sin\theta\,, \qquad (3.8)$$

$$\mathbf{a}_\phi = -\mathbf{i}\sin\phi + \mathbf{j}\cos\phi\,.$$

The Legendre polynomials satisfy the second-order differential equation

$$0 = \frac{d^2}{d\theta^2} P_n(\cos\theta) + \cot\theta \frac{d}{d\theta} P_n(\cos\theta) + n(n+1) P_n(\cos\theta)\,. \qquad (3.9)$$

It will be assumed that the medium is the vacuum, and that all dielectric and conduction effects are interpreted through secondary currents that will be explicitly computed during the calculations. Furthermore, it will be assumed that the dielectric effects can be neglected in this analysis, and that the conduction effects can be incorporated into the source-current distribution through Eq. (2.2).

With these assumptions, Maxwell's equations take the form

$$\nabla \times \mathbf{H} = \mathbf{K} + \sigma\mathbf{E} + \epsilon_0 \frac{\partial}{\partial t}\mathbf{E}\,, \qquad (3.10a)$$

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial}{\partial t}\mathbf{H}\,, \qquad (3.10b)$$

in MKS units. When the primary current $\mathbf{K}$ is given an expansion of the form (3.2), and when $\mathbf{E}$ and $\mathbf{H}$ are represented

by (3.2) and (3.3), substitution into (3.10) leads to the three scalar equations[2]:

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\, H^{0}_{C,n}(r,t)\right] = K^{0}_{B,n}(r,t) + \sigma(r,t)\, E^{0}_{B,n}(r,t) + \epsilon_0 \frac{\partial}{\partial t} E^{0}_{B,n}(r,t)\ , \qquad (3.11)$$

$$\frac{1}{r}\sqrt{n(n+1)}\, H^{0}_{C,n}(r,t) = K^{0}_{r,n}(r,t) + \sigma(r,t)\, E^{0}_{r,n}(r,t) + \epsilon_0 \frac{\partial}{\partial t} E^{0}_{r,n}(r,t)\ , \qquad (3.12)$$

$$\mu_0 \frac{\partial}{\partial t} H^{0}_{C,n}(r,t) = \frac{1}{r}\frac{\partial}{\partial r}\left[r\, E^{0}_{B,n}(r,t)\right] - \frac{1}{r}\sqrt{n(n+1)}\, E^{0}_{r,n}(r,t)\ . \qquad (3.13)$$

When successive integer values of n are substituted, each mode is then described by a set of three coupled first-order differential equations relating the unknown field components, $E^{0}_{r,n}$, $E^{0}_{B,n}$, $H^{0}_{C,n}$, to the (presumed to be known) primary current components, $K^{0}_{r,n}$ and $K^{0}_{B,n}$, and the (presumed to be known) conductivity function, $\sigma(r,t)$.

Equations (3.11-13) are a transformed version of Maxwell's equations (3.10), and therefore admit both retarded and advanced solutions, or arbitrary linear combinations of retarded and advanced solutions. The physical causality requirement is not contained in (3.11-13), and must be specified through an auxiliary condition, if mathematical solutions of (3.11-13) are to represent actual phenomena.

4. RETARDED HERTZ VECTOR

A solution to Maxwell's equations, and in particular a solution to the scalar equations (3.11-13), can be based on the retarded Hertz vector. In this way the limitation to the retarded solution, with the complete exclusion of the advanced solution, is fully enforced.

In terms of the Hertz vector, $\underset{\sim}{\Pi}$, the electric and magnetic field vectors are expressed through[6]:

$$\underset{\sim}{E} = \underset{\sim}{\nabla}\,\underset{\sim}{\nabla}\cdot\underset{\sim}{\Pi} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\underset{\sim}{\Pi} , \tag{4.1}$$

$$\underset{\sim}{H} = \epsilon_0 \underset{\sim}{\nabla}\times\frac{\partial}{\partial t}\underset{\sim}{\Pi} . \tag{4.2}$$

The retarded Hertz vector is a function of the actual source-current distribution, $\underset{\sim}{J}$, as given in (2.2) for the special symmetry which is being considered here. In the general case, with the source-current density $\underset{\sim}{J}(\underset{\sim}{r}_1, t_1)$ given the full expansion, involving the vector spherical harmonics $\underset{\sim}{P}{}_n^m$, $\underset{\sim}{B}{}_n^m$, and $\underset{\sim}{C}{}_n^m$, the retarded Hertz vector[3] separates into modes corresponding to the modes for the source current. For each mode, the Hertz-vector expansion coefficients are given in terms of the source-current expansion coefficients by the matrix equation:

$$\Pi^{m}_{\sigma,n}(r,t) = \frac{1}{2\epsilon_o}\int_{r_1=0}^{r_1=\infty}(r_1/r)\,dr_1\, G^{(n)}_{\sigma,\lambda}(r,r_1,r+r_1)\int_{t_1=t_o}^{t_1=t-(r+r_1)/c} J^{m}_{\lambda,n}(r_1,t_1)\,dt_1$$

$$- \frac{1}{2\epsilon_o}\int_{r_1=0}^{r_1=r}(r_1/r)\,dr_1\, G^{(n)}_{\sigma,\lambda}(r,r_1,r-r_1)\int_{t_1=t_o}^{t_1=t-(r-r_1)/c} J^{m}_{\lambda,n}(r_1,t_1)\,dt_1$$

$$- \frac{1}{2\epsilon_o}\int_{r_1=r}^{r_1=\infty}(r_1/r)\,dr_1\, G^{(n)}_{\sigma,\lambda}(r,r_1,r_1-r)\int_{t_1=t_o}^{t_1=t-(r_1-r)/c} J^{m}_{\lambda,n}(r_1,t_1)\,dt_1$$

$$+ \frac{1}{2\epsilon_o}\int_{r_1=0}^{r_1=r}(r_1/r)\,dr_1\int_{t_1=t-(r+r_1)/c}^{t_1=t-(r-r_1)/c} G^{(n)}_{\sigma,\lambda}(r,r_1,ct-ct_1)\; J^{m}_{\lambda,n}(r_1,t_1)\,dt_1$$

$$+ \frac{1}{2\epsilon_o}\int_{r_1=r}^{r_1=\infty}(r_1/r)\,dr_1\int_{t_1=t-(r+r_1)/c}^{t_1=t-(r_1-r)/c} G^{(n)}_{\sigma,\lambda}(r,r_1,ct-ct_1)\; J^{m}_{\lambda,n}(r_1,t_1)\,dt_1 \quad . \qquad (4.3)$$

The Greek indices, σ and λ, run through the three values: r, B, C. A repeated Greek index indicates summation over these three values. Thus (4.3) represents three equations, according as σ represents r, B, or C.

In Eq. (4.3) the Green's functions $G^{(n)}_{\sigma,\lambda}(r,r_1,s)$ are odd functions of the parameter <u>s</u>, and can be defined through

$$G^{(n)}_{\sigma,\lambda}(r,r_1,s) = \int_{s=0}^{s=s} g^{(n)}_{\sigma,\lambda}(\zeta)\, ds , \tag{4.4}$$

where the variable ζ is defined by

$$\zeta = \frac{1}{2\, r\, r_1}(r^2 + r_1^2 - s^2) , \tag{4.5}$$

and the nine functions $g^{(n)}_{\sigma,\lambda}(\zeta)$ are:

$$g^{(n)}_{r,r}(\zeta) = \zeta\, P_n(\zeta) , \tag{4.6a}$$

$$g^{(n)}_{B,B}(\zeta) = \zeta\, P_n(\zeta) + \frac{(1-\zeta^2)}{n(n+1)} \frac{d}{d\zeta} P_n(\zeta) , \tag{4.6b}$$

$$g^{(n)}_{C,C}(\zeta) = P_n(\zeta) , \tag{4.6c}$$

$$g^{(n)}_{r,B}(\zeta) = g^{(n)}_{B,r}(\zeta) = \frac{(1-\zeta^2)}{\sqrt{n(n+1)}} \frac{d}{d\zeta} P_n(\zeta) , \tag{4.6d}$$

$$g^{(n)}_{r,C}(\zeta) = g^{(n)}_{C,r}(\zeta) = 0 , \tag{4.6e}$$

$$g^{(n)}_{B,C}(\zeta) = g^{(n)}_{C,B}(\zeta) = 0 . \tag{4.6f}$$

In the integrals in (4.3), the time t_o is chosen to precede the initiation of the transient current flow, which is responsible for the generation of the field distribution which is incorporated into the Hertz vector (4.3). Any pre-existing static electric field can be expressed as an earlier flow of current, producing the distribution of electric charge which is to be associated with this initial electrostatic charge distribution. Thus if the initial moment, t_o, is pushed back far enough in the time scale, the Hertz vector (4.3) will incorporate all transient effects and any pre-existing static effects.

Once the retarded Hertz vector (4.3) has been expressed in terms of a particular source-current distribution, the corresponding electric and magnetic field vectors can be calculated directly through (4.1-2), and it is found that the mode separation is retained. For the particular symmetry described by (3.11-13), the three nonvanishing electromagnetic field components are all found to have explicit representations of the same general form as (4.3). Each of the components $E^o_{r,n}(r,t)$, $E^o_{B,n}(r,t)$, $H^o_{C,n}(r,t)$, is expressed as a sum of several integrals containing $J^o_{r,n}(r_1,t_1)$ or $J^o_{B,n}(r_1,t_1)$ in the integrand, with the integration taken over the causally accessible region of the (r_1,t_1)-plane.[3]

5. RETARDED FIELDS FOR FIRST MODE

The mode for which $n = 0$, which will be called the 'zeroth mode,' involves only the radial component of the electric field and the radial component of the current density. The equation that relates these radial components is

$$E^{0}_{r,0}(r,t) = -\frac{1}{\epsilon_0}\int_{t_1=t_0}^{t_1=t} J^{0}_{r,0}(r,t_1)\,dt_1 , \qquad (5.1)$$

which is equivalent to Gauss's law. For this mode the effects of retardation are present in the retarded Hertz vector, but drop out when the electric field (5.1) is computed from the Hertz vector.

However, retardation effects remain in the field components that are computed for the mode with $n = 1$, which will be called the 'first mode.' The distinction between causal and non-causal solutions of Maxwell's equations can thus be illustrated by the fields associated with this first mode. It will be assumed, as stated earlier, that the circulatory current component, $J^{0}_{C,1}$, is zero, so that only the field components $E^{0}_{r,1}$, $E^{0}_{B,1}$, and $H^{0}_{C,1}$ are generated. The retarded fields, obtained from the retarded Hertz vector, will then be written explicitly, with the aid of the abbreviation:

$$(\iint) = \int_{r_1=0}^{r_1=r} \int_{t_1=t-(r+r_1)/c}^{t_1=t-(r-r_1)/c} + \int_{r_1=r}^{r_1=\infty} \int_{t_1=t-(r+r_1)/c}^{t_1=t-(r_1-r)/c} , \quad (5.2a)$$

which can also be written in the equivalent form (but with the order of integration reversed):

$$(\iint) = \int_{t_1=-\infty}^{t_1=t-r/c} \int_{r_1=-r+c(t-t_1)}^{r_1=r+c(t-t_1)} + \int_{t_1=t-r/c}^{t_1=t} \int_{r_1=r-c(t-t_1)}^{r_1=r+c(t-t_1)} . \quad (5.2b)$$

In (5.2b) the limit $t_1=-\infty$ will ordinarily be replaced by $t_1=t_o$, where t_o is a time preceding any of the transient current flow.

The radial component of (4.1), for this first mode, gives

$$E^0_{r,1}(r,t) = -\frac{1}{\epsilon_o} \int_{t_1=t_o}^{t_1=t} J^0_{r,1}(r,t_1)\, dt_1$$

$$+ \frac{2}{3\epsilon_o} \int_{r_1=0}^{r_1=r} \int_{t_1=t_o}^{t_1=t-(r-r_1)/c} (r_1^2/r^3) \left[J^0_{r,1}(r_1,t_1) + \sqrt{2}\, J^0_{B,1}(r_1,t_1)\right] dt_1 dr_1$$

$$+ \frac{\sqrt{2}}{3\epsilon_o} \int_{r_1=r}^{r_1=\infty} \int_{t_1=t_o}^{t_1=t-(r_1-r)/c} (1/r_1) \left[\sqrt{2}\, J^0_{r,1}(r_1,t_1) - J^0_{B,1}(r_1,t_1)\right] dt_1\, dr_1$$

$$- \frac{1}{3\epsilon_o} (\textstyle\iint) \left[(r_1^2/r^3) + (1/r_1)\right] J^0_{r,1}(r_1,t_1)\, dt_1\, dr_1$$

$$- \frac{\sqrt{2}}{6\epsilon_o} (\textstyle\iint) \left[2(r_1^2/r^3) - (1/r_1)\right] J^0_{B,1}(r_1,t_1)\, dt_1\, dr_1$$

$$+ \frac{1}{3\epsilon_o} (\textstyle\iint) (1/r^3 r_1) \left[3(r^2+r_1^2)c(t-t_1) - c^3(t-t_1)^3\right] J^0_{r,1}(r_1,t_1)\, dt_1\, dr_1$$

$$- \frac{\sqrt{2}}{6\epsilon_o} (\textstyle\iint) (1/r^3 r_1) \left[3(r^2-r_1^2)c(t-t_1) - c^3(t-t_1)^3\right] J^0_{B,1}(r_1,t_1)\, dt_1\, dr_1. \qquad (5.3)$$

The component of Eq. (4.1) which is in the polar direction $\underset{\sim}{a}_\theta$ gives the result, for this first mode:

$$E^0_{B,1}(r,t) = \frac{1}{2\epsilon_o} \int_{r_1=0}^{r_1=\infty} (r_1/cr)\, J^0_{B,1}(r_1, t-[r+r_1]/c)\, dr_1$$

$$- \frac{1}{2\epsilon_o} \int_{r_1=0}^{r_1=r} (r_1/cr)\, J^0_{B,1}(r_1, t-[r-r_1]/c)\, dr_1$$

$$- \frac{1}{2\epsilon_o} \int_{r_1=r}^{r_1=\infty} (r_1/cr)\, J^0_{B,1}(r_1, t-[r_1-r]/c)\, dr_1$$

$$
- \frac{\sqrt{2}}{3\epsilon_o} \int_{r_1=0}^{r_1=r} \int_{t_1=t_o}^{t_1=t-(r-r_1)/c} (r_1^2/r^3) \left[J^0_{r,1}(r_1,t_1) + \sqrt{2}\; J^0_{B,1}(r_1,t_1) \right] dt_1\, dr_1
$$

$$
+ \frac{2}{3\epsilon_o} \int_{r_1=r}^{r_1=\infty} \int_{t_1=t_o}^{t_1=t-(r_1-r)/c} (1/r_1) \left[\sqrt{2}\; J^0_{r,1}(r_1,t_1) - J^0_{B,1}(r_1,t_1) \right] dt_1\, dr_1
$$

$$
+ \frac{\sqrt{2}}{6\epsilon_o} (\int\int) \left[(r_1^2/r^3) - 2\,(1/r_1) \right] J^0_{r,1}(r_1,t_1)\; dt_1\; dr_1
$$

$$
+ \frac{1}{3\epsilon_o} (\int\int) \left[(r_1^2/r^3) + (1/r_1) \right] J^0_{B,1}(r_1,t_1)\; dt_1\; dr_1
$$

$$
+ \frac{\sqrt{2}}{12\epsilon_o} (\int\int) (1/r^3 r_1) \left[3(r^2-r_1^2)c(t-t_1) + c^3(t-t_1)^3 \right] J^0_{r,1}(r_1,t_1) dt_1 dr_1
$$

$$
- \frac{1}{12\epsilon_o} (\int\int) (1/r^3 r_1) \left[3(r^2+r_1^2)c(t-t_1) + c^3(t-t_1)^3 \right] J^0_{B,1}(r_1,t_1) dt_1 dr_1 \tag{5.4}
$$

The nonvanishing component of Eq. (4.2), which is in the azimuthal direction $\underset{\sim}{a}_\phi$, gives the corresponding equation for the nonvanishing component of the magnetic field:

$$H^{0}_{C,1}(r,t) = -\frac{1}{2}\int_{r_1=0}^{r_1=\infty} (r_1/r)\, J^{0}_{B,1}(r_1, t-[r+r_1]/c)\, dr_1$$

$$+\ \frac{1}{2}\int_{r_1=0}^{r_1=r} (r_1/r)\, J^{0}_{B,1}(r_1, t-[r-r_1]/c)\, dr_1$$

$$-\ \frac{1}{2}\int_{r_1=r}^{r_1=\infty} (r_1/r)\, J^{0}_{B,1}(r_1, t-[r_1-r]/c)\, dr_1$$

$$+\ \frac{\sqrt{2}\,c}{4}\,(\int\int)\ (1/r^2 r_1)\left[(r^2+r_1^{2}) - c^2(t-t_1)^2\right] J^{0}_{r,1}(r_1,t_1)\, dt_1\, dr_1$$

$$-\ \frac{c}{4}\,(\int\int)\ (1/r^2 r_1)\left[(r^2-r_1^{2}) - c^2(t-t_1)^2\right] J^{0}_{B,1}(r_1,t_1)\, dt_1\, dr_1\ . \tag{5.5}$$

It can be verified through explicit substitution that the three field components for this first mode, as given in (5.3-5), satisfy the reduced equations (3.11-13), if the source current is given by (2.2).

6. NUMERICAL SOLUTION OF INTEGRAL EQUATION

The integral equation to be solved is actually a coupled pair of equations for the current-density components,

$$J^{0}_{r,1}(r,t) = K^{0}_{r,1}(r,t) + \sigma(r,t)\, E^{0}_{r,1}(r,t) , \qquad (6.1a)$$

$$J^{0}_{B,1}(r,t) = K^{0}_{B,1}(r,t) + \sigma(r,t)\, E^{0}_{B,1}(r,t) , \qquad (6.1b)$$

together with the equations (5.3) and (5.4) which give the electric field components in terms of integrations over the current-density components.

In the problem of direct interest[4], the primary current $\underset{\sim}{K}$ and the transient conductivity function σ are both initiated by the nuclear reaction, and are therefore zero before the time of the detonation. At a distance r from the detonation point, $\underset{\sim}{K}$ and σ will be zero up until a time which is later than the detonation time by the amount r/c, the propagation time for gamma rays moving radially outward from the detonation. If the detonation occurs at $t = 0$, then the problem domain in the (r,t)-plane is limited to the region to the right of a diagonal line through the origin, as shown in Figure 1.

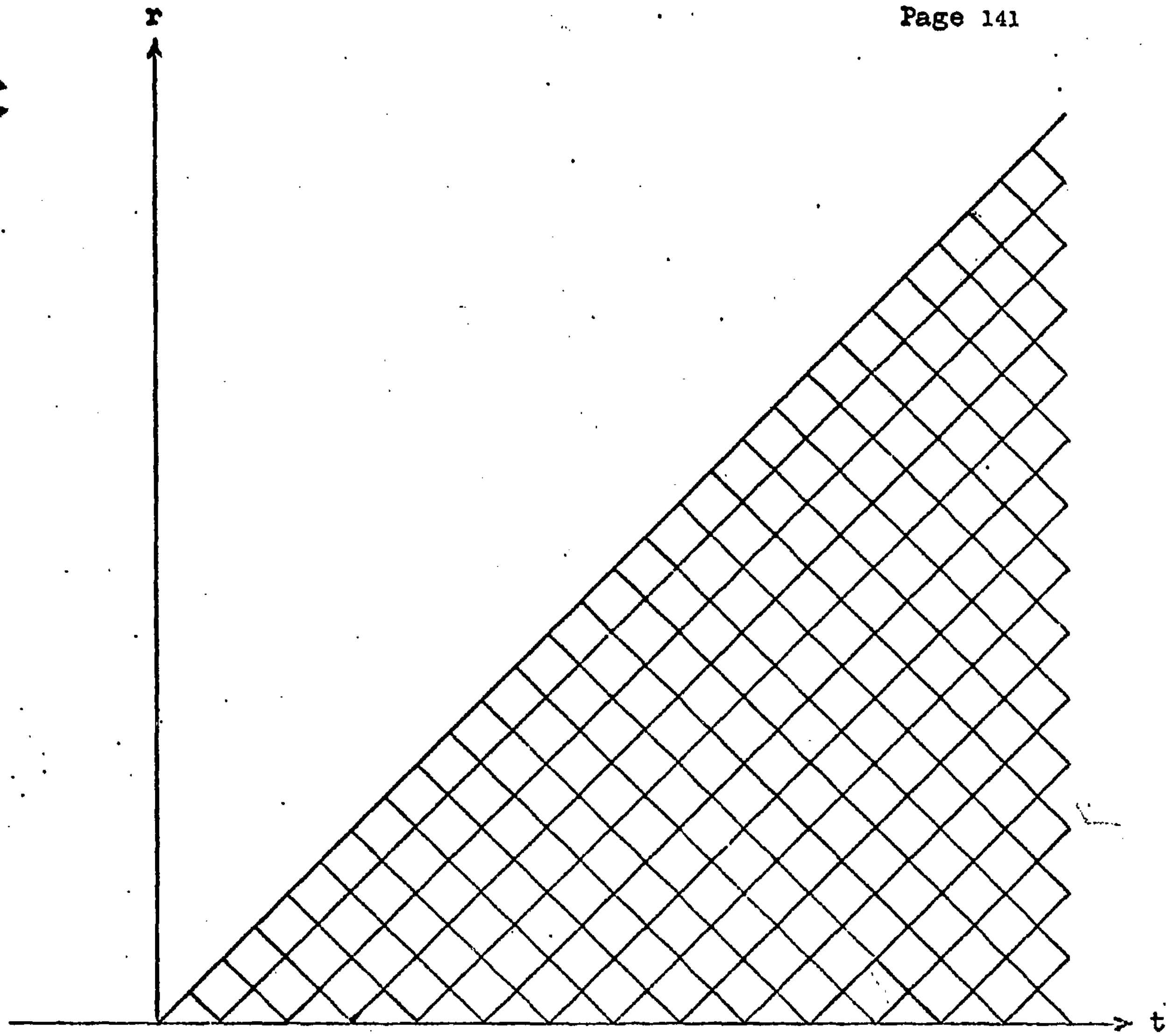

Figure 1. The problem domain, subdivided by a diagonal lattice.

For the numerical solution, the problem domain is subdivided by a diagonal lattice, bounded on the left by the line

$$r = ct , \tag{6.2a}$$

and bounded below by the line

$$r = 0 . \tag{6.2b}$$

Field and current component values are calculated for the points where lattice lines cross. Integrations over previously-calculated current values are expressed as summations over the small square areas in the lattice.

The lattice-line spacing is made sufficiently fine so that the error introduced through the discreteness is small. In practice, the line spacings can differ in different portions of the problem domain, and the small areas can be rectangular instead of square, if necessary to make efficient use of computing-machine time and storage. At the domain boundary (6.2a), in particular, the lattice spacing should be small enough so that, in the problem being analyzed[4], the secondary current, $\sigma \underset{\sim}{E}$, is still very small in comparison with the primary current, $\underset{\sim}{K}$. The initial calculations of the field components can then be based on the known primary current. As the computation

progresses from one lattice point to the next, and moves to regions that are not close to the boundary (6.2a), the secondary current will play an increasingly important role.

The numerical solution also requires initial current-density values along the inner boundary (6.2b). In the problem being considered[4], this boundary refers to the immediate neighborhood of the nuclear detonation, for times subsequent to the detonation time. The asymmetry which leads to the first-mode portion of the primary current is the asymmetry produced by the presence of the ground or by the presence of an atmospheric air-density gradient. In practice this asymmetry will not enter until at least a short distance from the detonation center. Thus the first-mode portion of the primary current can be set equal to zero along the inner boundary (6.2b), in this application of the theory. For other applications, physical considerations or a simple analytical model will generally give an adequate basis for setting the initial current-density values along the boundary (6.2b).

The numerical iteration can proceed ~~proceeds~~ diagonally upward to the right, in Fig. 1, along lattice lines parallel to the diagonal boundary (6.2a). One step in the iteration is indicated in Figure 2. Here the current components (6.1)

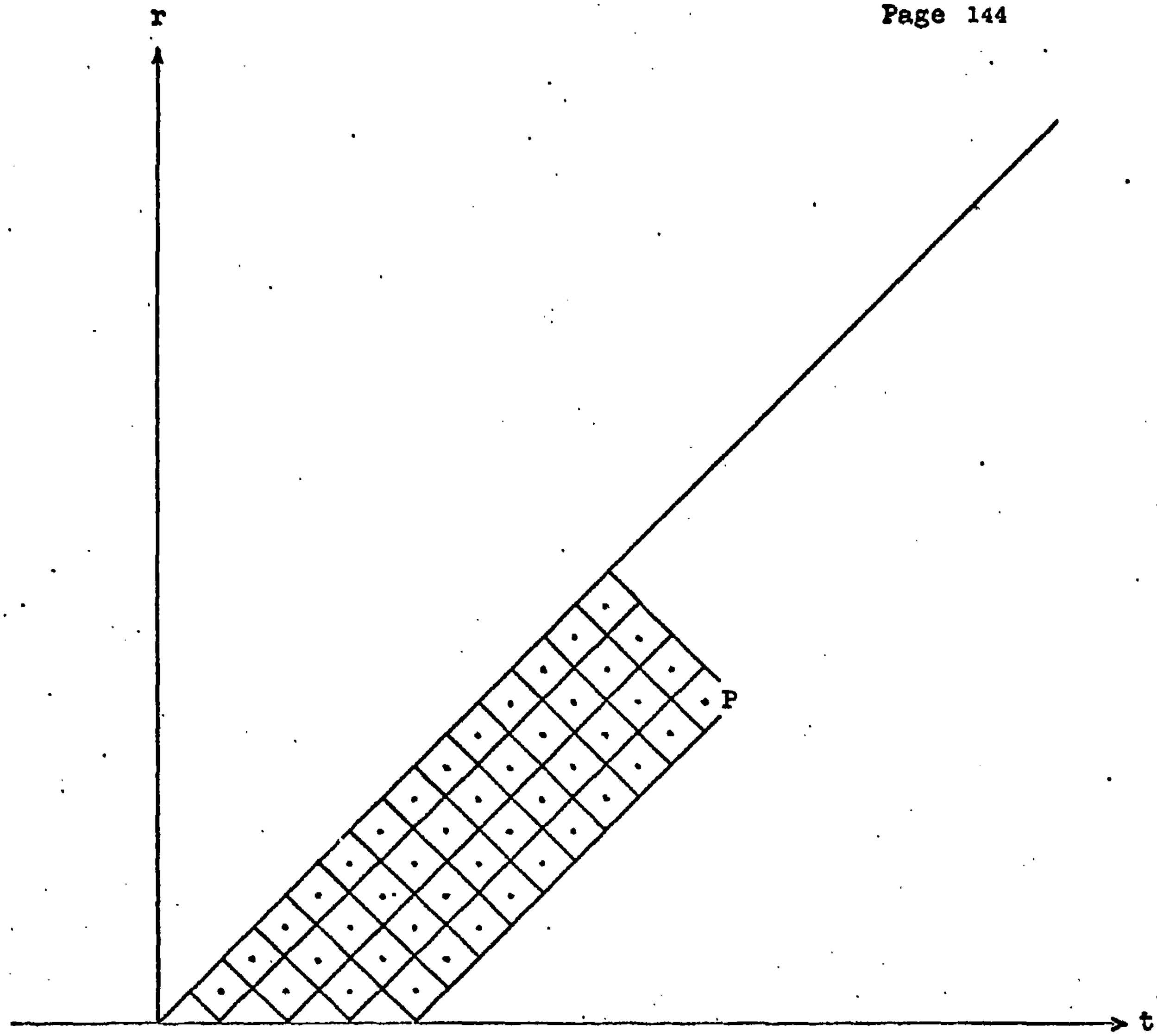

Figure 2. A causal iteration step.

are being computed at the lattice point P, with the integrations (5.3) and (5.4) replaced by summations over the lattice squares shown in the figure. Each square is treated as though it were concentrated into a point source having the location of its central point, shown here as a dot in the center of the square. The current associated with the square is the average of the current values at its corners, previously computed. (The triangles at the bottom can be included separately.)

In the case of the square which has the point P at its right-hand corner, the unknown current at P can be approximated through extrapolation from the three currents at the other corners. Alternatively, the current at P can be included formally in the summation (to which the integration has been reduced by this approximate method), then transferred algebraically to the left-hand side of (6.1), and solved for as a part of the numerical solution for the current-density components at P.

Either alternative will give the current at P, and the iteration can then step to the next point on the lattice. The final result will be a set of current-component values for all the lattice points. If desired, the program can then be re-run, with these values substituted when the current at P is needed for the integral over the square containing the point P as its right-hand corner. If the lattice is fine enough, the change in current values, resulting from this re-run, should be small.

The numerical solution which has been described will give the current values at the lattice points, when the primary current $\underline{K}$ and the conductivity σ are inserted, along with the boundary values for the current on the inner boundary (6.2b). The transient electric and magnetic field components will also be given, through integration of this current distribution, using (5.3-5). The result will be consistent with causality requirements, provided the lattice is sufficiently fine. In practice, if the solution is not changed significantly when the lattice is made still finer, then it was already sufficiently fine.

While this illustration has referred to the first mode, the mode with $n = 1$, the same general considerations apply to the solution of the integral equation for any higher mode, as long as the modes are separable. If the conductivity function is not given by $\sigma(r,t)$, but by a function which depends upon θ as well, then the modes will be coupled together and (6.1) will be replaced by an extended matrix equation, coupling together the current components for the different modes. In this case, a generalized version of the iteration described here can still be used, without violating the causality requirements.

7. NON-CAUSAL SOLUTION

For comparison, a non-causal solution[2] of Maxwell's equations, in the reduced form given in (3.11-13), will be described briefly. In this non-causal solution, the same problem domain of Fig. 1 has been used, but the lattice has the orientation shown here in Figure 3. The iteration progresses vertically upward from the boundary (6.2b), and requires initial values along this boundary and also along and near the boundary (6.2a).

An iteration step is shown in Figure 4. The field components are calculated at the point P, based on the primary current and the conductivity at the point X and the previously-calculated field components at the three points Q. A finite-difference version of the equations (3.11-13) is used in this calculation. The result is a numerical solution of Maxwell's equations, for each step, but each step gives a mixing of advanced and retarded solutions since the causality requirements have not been imposed in any way. Thus the result is a mathematical solution of Maxwell's equations which has a doubtful relationship to physical phenomena.

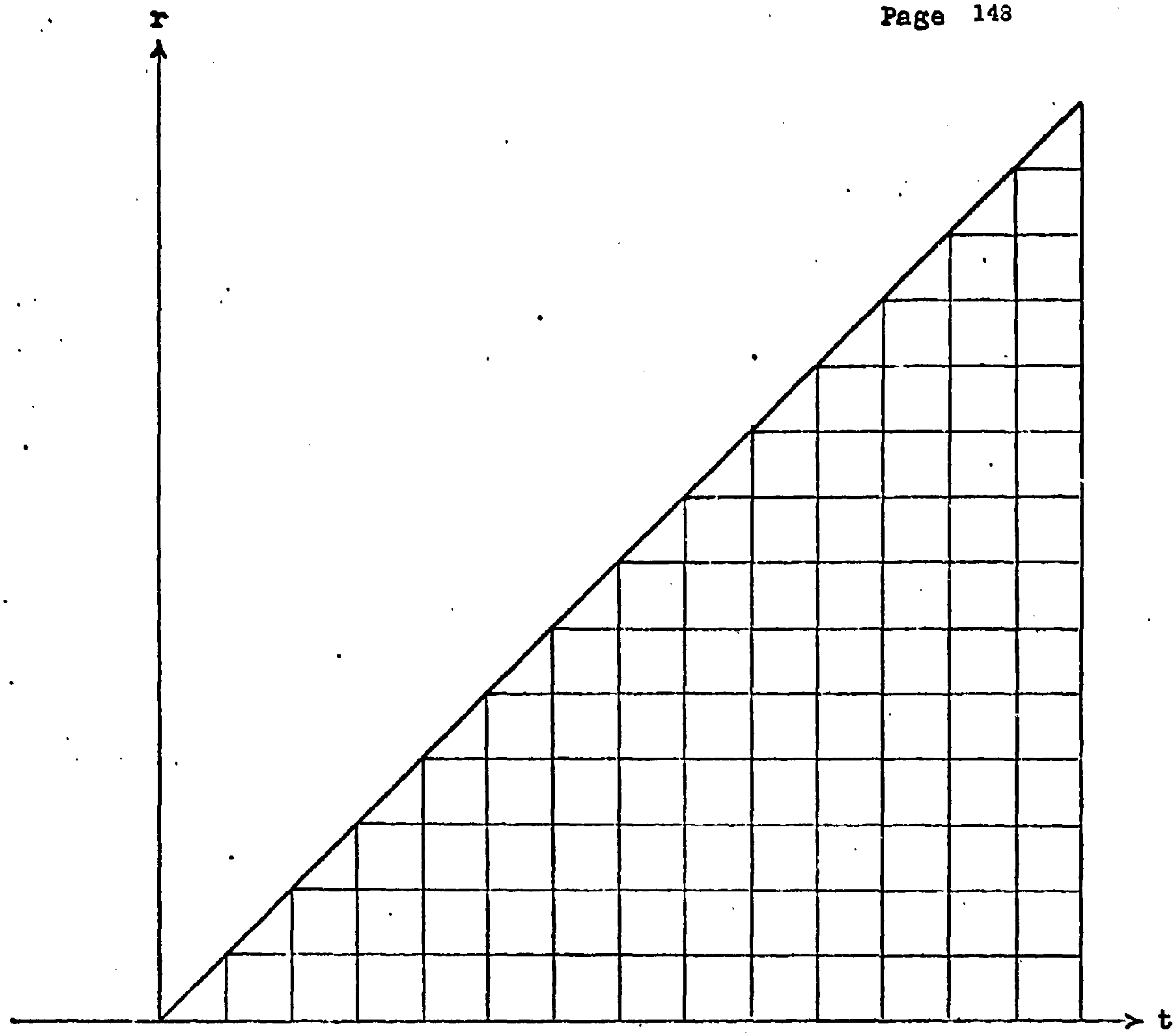

Figure 3. The problem domain, subdivided by a rectangular lattice.

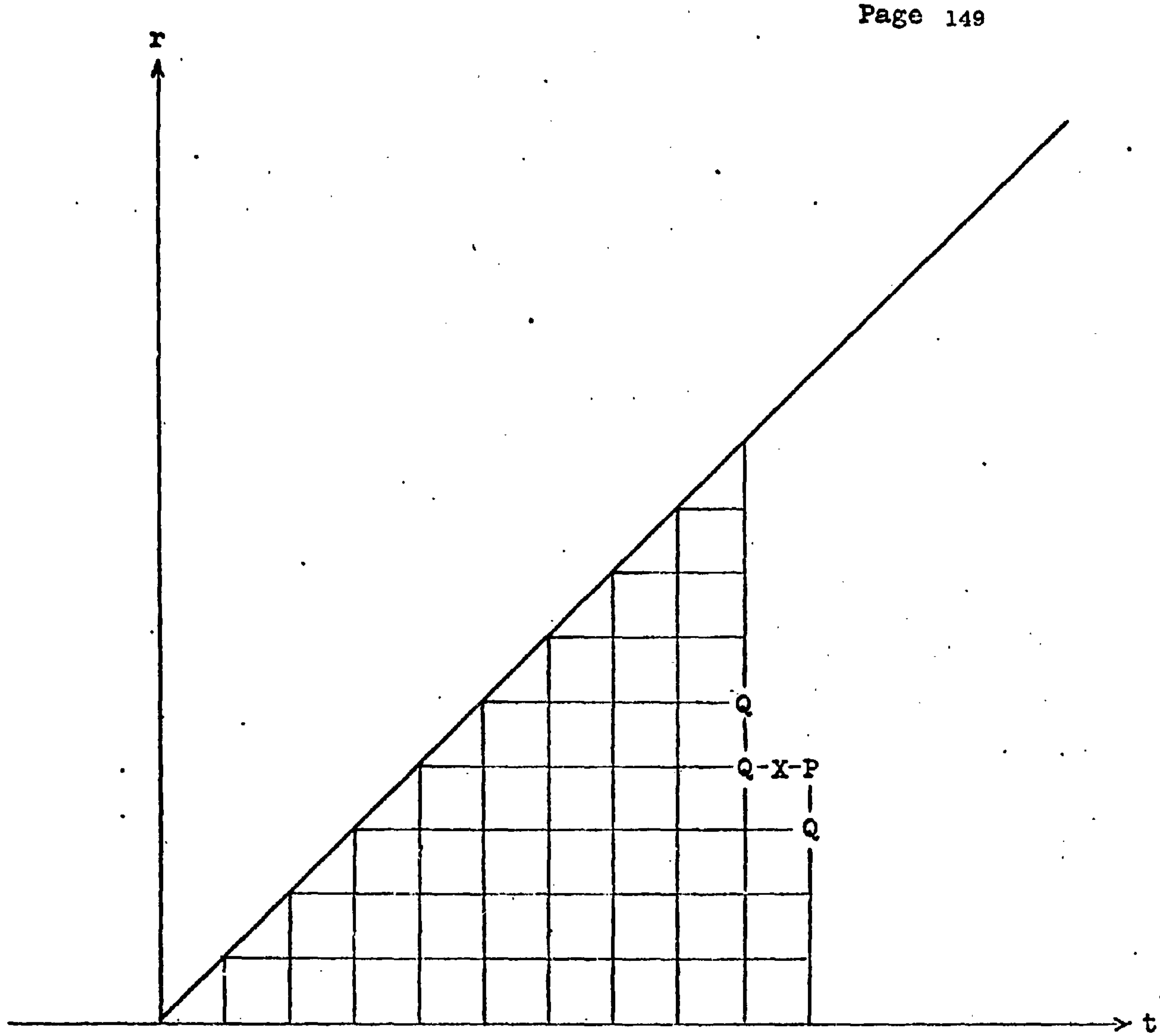

Figure 4. A non-causal iteration step.

It can be seen from Fig. 4 that the introduction of an intensification of the primary current at the point X that is shown will have effects on point P, as shown, and hence also on other points P which lie directly above this one in the figure. Thus a current change at an inner radius will lead to field changes at an outer radius which are essentially simultaneous, and must have traveled faster than c. These effects are physically inadmissible , yet they do satisfy the finite-difference form of Maxwell's equations which has been used in the iteration step of Fig. 4.

The solution, of course, is not to change to another form of iteration step which just translates Eqs. (3.11-13) from differential form to difference form in some other way. The solution, obviously, to the problem of solving Maxwell's equations numerically in the (r,t)-plane, is to impose the causality requirements at an earlier stage, when the equations are in four-dimensional space-time, and to select the retarded solution at the beginning. The reduction to the (r,t)-plane then leads to the retarded Hertz vector (4.3) and the retarded fields illustrated in (5.3-5).

8. CONCLUSIONS

In the general case, a numerical solution of Maxwell's equations will give a superposition of electromagnetic phenomena which can occur in a physical situation, and other phenomena which cannot occur physically. If the mathematical solution is to describe a physical process, then the analysis must be set up in such a way that only the retarded-field solution of Maxwell's equations is admitted, while the advanced fields are excluded from the beginning.

In problems where there is a center of symmetry or of partial symmetry, a description of the problem in terms of spherical polar coordinates is appropriate. In certain problems an expansion in vector spherical harmonics provides a separation into modes, and a reduction of the four-dimensional problem into a set of two-dimensional problems, one for each mode. In the reduced problem, the variables are the radial distance r and the time t.

If retardation is ignored, then a non-causal solution in the (r,t)-plane can be obtained through a point-to-point numerical integration of a finite-difference version of Maxwell's equations, as they appear when reduced to two dimensions by the separation into vector-spherical-harmonic

modes. This solution will in the general case violate the causality requirement and must therefore be ruled out on physical grounds.

When retardation is incorporated at the beginning of the analysis, before the separation of variables, then the physical causality requirement will be satisfied. The reduction to the (r,t)-plane can still be carried out, but in this case the result is an integral equation to be solved numerically, rather than a differential equation. The solution of the integral equation will provide one of the possible solutions of the differential equation, but will exclude the non-physical solutions.

The integral equation can be solved numerically with the aid of a discrete lattice in the (r,t)-plane. Each iteration step then involves a summation over all the causally-accessible lattice positions in the region covered by previous iteration steps. The iteration procedure resembles in a mechanical way the iteration used in the non-causal finite-difference solution, so that the conversion of a computing-machine program from the non-causal solution to the causal solution can readily be carried out.

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